§1 k local field (or algebraic number field)

classical simply connected (absolute almost) simple algebraic

k-gps.

Type ${}^1A_n$ $\qquad G = SL_m(D)$

D: central division algebra / central skew field over k of index d

$n = md - 1$

$\{x \in M_m(D) \mid Nrd(x) = 1\}$

Type ${}^2A_n$ $\qquad G = SU_{n+1}(L, f)$

L: quadratic extension of k

f: nondegenerate form on L^{n+1}

Hermitian relative to $\sigma \in Gal(L/k)$ non-trivial.

[Over a local field there is no skew field of index $d>1$ having an involution of second kind]

$$f(x_1, \cdots x_n ; y_1, \cdots y_n) = a_1 \sigma(x_1) y_1 + \cdots + a_n \sigma(x_n) y_n$$

$a_i \in K$.

Type C_m :

i) $G = Sp_{2m}(K, f)$ f: non-degenerate alternating form on K $\quad g^* = (\bar{g}_{ji})$

ii) $G = SU_m(D, f) = \{ g \in GL_m(D) \mid g^* F g = g \quad Nrd_{M_m(D)/K}(g) = 1 \}$

$\boxed{F^* = F}$

D quaternion div alg $/ K$

f: nondegenerate Hermitian form (relative to the standard involution on D) on an m-dim'l space W over D

$$f(x_1, \cdots x_n ; y_1, \cdots y_n) = I(x_1) a_1 y_1 + \cdots + I(x_n) a_n y_n$$

$a_i \in K$.

Type B_n

universal cover of $SO_{2n+1}(k, f)$: $\mathrm{spin}_{2n+1}(f)$

f : symmetric non-deg bilinear form on k

Type D_n

$\mathrm{spin}_{2n}(k, f)$ f : symmetric non-deg bilinear form on k

universal cover of

$SU_n(D, f)$

D quaternion div alg $/ k$

f : nondegenerate skew-Hermitian form (relative to the standard involution on D) on an n-diml space W over D

(Discriminant $= 1$ $\quad {}^1D_n$

$\neq 1$ $\quad {}^2D_n$)

§2 Classifying bilinear forms

K : finite Galois $/k$ $\quad Gal(K/k) = \Gamma_K$

$\{k\text{-vector spaces}\} \cong \{K\text{-vector spaces}$

with semi-linear action of $\Gamma_K\}$

$$V \mapsto V_K = V \otimes_k K$$

$$V^{\Gamma_K} \leftarrow V$$

<u>semi-linear action</u> $\quad \Gamma_K \to \mathrm{Aut}_{k\text{-linear}}(V)$

$$\sigma(cv) = \sigma(c)\,\sigma v \qquad \sigma \in \Gamma_K \quad c \in K, \ v \in V$$

V_0: k-vector space

$\phi_0 : V_0 \times V_0 \to k$ bilinear form

$(V_0, \phi_0)_K$ ext by scalars

$A(K) = \mathrm{Aut}((V_0, \phi_0)_K)$

$\alpha : (V_0)_K \to (V_0)_K$ $\quad K$-linear

s.t. $(\phi_0)_K(\alpha v, \alpha w) = \phi_{0K}(v, w)$ $\quad \forall v, w \in (V_0)_K$

Thm $H^1(\Gamma_K, A(K))$ classifies isom classes of (V, ϕ) over k s.t $(V,\phi)_K \cong (V_0, \phi_0)_K$

Pf $f: (V_0, \phi_0)_K \to (V, \phi)_K$ isom

$$a_\sigma = f^{-1} \circ \sigma f$$

Conversely if (a_σ) is a 1-cocycle

Let $V_{0K} = K \otimes_k V_0$

Define Γ-action on $(V_0)_K$ via

$$^\sigma x = a_\sigma \cdot \sigma x \qquad \sigma \in \Gamma \quad v \in (V_0)_K$$

Then $\sigma(cx) = a_\sigma \cdot \sigma(cx) = \sigma(c)\, ^\sigma x$

$$^{\sigma\tau}x = a_{\sigma\tau}\, \sigma\tau(x)$$

$$^\sigma(^\tau x) = a_\sigma \cdot \sigma(^\tau x) = a_\sigma \cdot \sigma(a_\tau \tau x)$$

$$= a_\sigma\, \sigma a_\tau\, \sigma(\tau x)$$

Let $V_1 = \{ x \in V_{0K} \mid {}^{\sigma}x = x \}$

Then $K \otimes_k V_1 \cong V_{0K}$ $\qquad x, y \in V_1$

$\Rightarrow \phi_{0K}({}^{\sigma}x, {}^{\sigma}y) = \phi_{0K}(a_\sigma \sigma x, a_\sigma \sigma y)$

$= \phi_{0K}(\sigma x, \sigma y) = \phi_{0K}(x, y)$

$= \sigma \phi_{0K}(x, y)$

$\Rightarrow \phi_{0K}(x, y) \in k$ $\qquad \square$

<u>Prop</u> $\forall n \quad H^1(\Gamma_K, GL_n(K)) = 1$.

<u>pf</u> $\quad GL_n(K) = \mathrm{Aut}(K^n) \qquad V_0 = k^n$

$\Rightarrow H^1(\Gamma_K, GL_n(K))$ classifies isom classes of k-vector spaces V s.t $V \otimes_k K \simeq K^n$ $\Rightarrow V \cong k^n$. $\square$

<u>Prop</u> $\forall n \quad H^1(\Gamma, SL_n(K)) = 1$.

<u>pf</u> We have

$$1 \to SL_n(K) \to GL_n(K) \xrightarrow{\det} K^\times \to 1$$

$$\Rightarrow \quad 1 \to SL_n(k) \to GL_n(k) \xrightarrow{\det} k^\times \to H^1(\Gamma, SL_n(k))$$

$\downarrow$ surj

$$\to H^1(\Gamma, GL_n(k))$$

$\Rightarrow$ ✓.

Prop ϕ_0: nondeg. alternating bilinear form on V_0, Sp associated symplectic gp

$H^1(\Gamma, Sp(k)) = 1$.

Pf It classifies isom classes of (V, ϕ)

All nondeg. bilinear alternating forms are equiv. /k.

$\square$

Prop The elts of $H^1(k, SO_n(Q))$ (quadr. form) are in one-to-one corr with the k-equiv. classes of those quadratic forms of degree n over k that have the same discriminant as Q.

§3 classifying the forms of an alg. gp

K/k finite Galois Γ_K

G_0 alg gp $/ k$

$A(K) = \mathrm{Aut}_K(G_K)$ (K-homo of k-objects)

$\Gamma \circlearrowright A(K)$ $\quad {}^{\sigma}\alpha = \sigma \alpha \sigma^{-1}$

<u>Thm</u> The cohomology set $H^1(\Gamma, A(K))$ Classifies the isom classes of algebraic gps G $/ k$ that become iso to G_0 over K

<u>Pf</u> G / k

$f : G_{0K} \to G_K$ isom.

let $a_\sigma = f^{-1} \circ {}^{\sigma} f$

Then (a_σ) is a 1-cocycle

<u>Surjectivity</u> $A(k) = \mathrm{Aut}(\mathcal{O}(G_{0K}))$

$$\mathcal{O}(G_{0K}) = K \otimes_k \mathcal{O}(G_0) \quad A_0 = \mathcal{O}(G_0)$$

$$A = K \otimes_k A_0$$

Define new action Γ_K on A via

$$\sigma_a = a_\sigma \circ \sigma$$

$$B = \{ a \in A \mid {}^\sigma a = a \}$$

Then $K \otimes_k B \to A$

Hopf alg structure on A induces Hopf structure on B

Let G_0 be an algebraic gp $/k$

The isom classes of alg gp $/k$ s.t

$G_{\bar{k}} \cong G_{0\bar{k}}$ are classified

by $H^1(\Gamma, A(\bar{k}))$.

Write $G^{ad} = G/Z(G)$.

Prop $\forall$ simply conneted s.s gp G,

$$1 \to G^{ad}(\bar{k}) \to \mathrm{Aut}_{\bar{k}}(G_{\bar{k}}) \to \mathrm{Sym}(D) \to 1$$

Dynkin diag

is an exact sequence.

When G is split, Γ acts trivially on

$\mathrm{Sym}(D)$, & the sequence is split,

i.e $\exists$ subgp of $\mathrm{Aut}_{\bar{k}}(G_{\bar{k}})$ on which

Γ acts trivially & maps isom. onto $\mathrm{Sym}(D)$.

§4 classical gps & algebras with involution.

Assume char $k=0$

<u>Def</u> A k-alg A is central if $Z(A)=k$, simple if it has no 2-sided ideals (except $0, A$)
If $a \in A-\{0\}$ invertible, A is divison alg / skew field)

<u>Prop</u> The isom classes of k-algebras becoming isom to $M_n(k)$ over $\bar{k}$ are classified by $H^1(k, PGL_n)$

<u>pf</u> All automorphisms of $M_n(k)$ are inner.

<u>The inner forms of SL_n</u>

Thm The inner forms of SL_n are the gps $SL_m(D)$ for D a division algebra of degree $\frac{n}{m}$.

$$SL_m(D) : R \mapsto \{ a \in M_m(R \otimes_k D) \mid Nm(a) = 1 \}$$

Pf Inner forms of SL_n are classified by $H^1(k, PGL_n)$, which also classifies forms of $M_n(k)$

The forms of $M_n(k)$ are $M_m(D)$.

$$SL_n \hookrightarrow M_n$$

$\square$

Algebras with involutions.

Let (A, τ) be a simple algebra with involution τ : $(\tau^2 = 1 \quad \tau(ab) = \tau(b)\tau(a))$

$Z(A) = L \qquad k = L^{\tau}$

a) (A, τ) is a simple k-algebra with involution of first kind, i.e. $Z(A) = k$ and $k^{\tau} = k$.

$$\Rightarrow \quad A \otimes_k \bar{k} \overset{\varphi}{\simeq} M_n(\bar{k})$$

We can choose φ s.t

$\tau \rightsquigarrow \quad (M_n(\bar{k}), \tau) \qquad \tau(x) = x^t$

or $(M_{2n}(\bar{k}), \tau) \qquad \tau(x) = J x^t J^{-1} \quad J = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$

b) (A, τ) is a simple k-algebra with involution of second kind, then $[L : k] = 2$

$$A \otimes_k \bar{k} = (A \otimes_k L) \otimes_L \bar{k} \simeq M_n(\bar{k}) \oplus M_n(\bar{k})$$

$$\tau \rightsquigarrow (M_n(\bar{k}) \oplus M_n(\bar{k}), \tau)$$

$$\tau(x, y) = (y^t, x^t).$$

(D, τ) divison algebra with involution τ

$A = M_m(D)$ $\quad a = (a_{ij})$ $\quad a^* := (\tau(a_{ji}))$

$*$: involution on A of same kind as τ.

$\sigma: A \to A \quad x \mapsto F^{-1} x^* F$

σ is an involution of A

over algebraic closure

$\sigma \rightsquigarrow \nu$ on $M_n(\bar{k})$

ν has the same type as τ if F is Hermitian

opposite —"— if F is skew-Herm.

τ: first type if $\dim_k D^\tau = \frac{n(n+1)}{2}$ $\Leftarrow$ over alg closure $x \to x^\tau$

second type if $= \frac{n(n-1)}{2}$ the other one.

The outer forms of SL_n

$$1 \to PGL_n(\bar{k}) \to \mathrm{Aut}(SL_{n\bar{k}}) \to \underbrace{\mathrm{Sym}\,D}_{\text{order } 2} \to 1$$

Consider

$(M_n(k) \times M_n(k), *)$ $\quad (X,Y)^* = (Y^t, X^t)$

Automorphisms of $(M_n(k) \times M_n(k), *)$ are inner aut by elts $(X, (X^t)^{-1})$ & compositions with $(X,Y) \mapsto (Y,X)$

Consider

$(*)$ $SL_n(\bar{k}) \hookrightarrow M_n(\bar{k}) \times M_n(\bar{k})$ $\quad X \mapsto (X, (X^t)^{-1})$

$\mathrm{Aut}(M_n(\bar{k}) \times M_n(\bar{k}), *) \simeq \mathrm{Aut}(SL_{n\bar{k}})$

$\Rightarrow$ forms of $SL_{n\bar{k}}$ $\longleftrightarrow$ forms of $(M_n(\bar{k}) \times M_n(\bar{k}), *)$

$\qquad\qquad\qquad\qquad\qquad\qquad$ $\|$

$\{$ simple algebra $/ k$ with centre $[K:k]=2$ and involution of second kind $\}$

Using the embedding (*), we see that

the form of $SL_n \leftrightarrow (A, *)$ is

$$R \mapsto G(R) = \{ a \in R \otimes_k A \mid a^* a = 1 \ \ Nm(a) = 1 \}$$

A: simple k algebra w/ inv of second kind

& centre a quadratic ext of k

$\Rightarrow A \simeq M_m(D)$ D division alg w/ inv ∂

$\downarrow$ of second kind $Z(D) = K$

involution θ $[K : k] = 2$

$$\theta|_L = \partial|_L$$

$$\theta(x) = F x^* F^{-1} \qquad F^* = F \qquad x^* = (\partial(x_{ji})) \quad x = (x_{ij})$$

The forms of Sp_{2n}

Consider

$(M_{2n}(k), *)$ $\quad X^* = SX^tS^{-1}$ $\quad S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

$(*)$ $Sp_{2n}(\bar{k}) \hookrightarrow M_{2n}(\bar{k}) \quad X \mapsto X$

$$\mathrm{Aut}(Sp_{2n\bar{k}}) \simeq \mathrm{Aut}(M_{2n}(\bar{k}), *)$$

$\mathrm{Int}(U)$ commutes with $* \Leftrightarrow U^*U = I$ (over $\bar{k}$)

$\Leftrightarrow SU^tS^{-1}U = I \Leftrightarrow U^tSU = S \Leftrightarrow U \in Sp_{2n\bar{k}}$

Forms of $Sp_{2n} \leftrightarrow$ Forms of $(M_{2n}(k), *)$

$(*)$ identifies $Sp_{2n}(\bar{k})$ with subgp of $M_{2n}(\bar{k})$

$a^*a = 1$

$\Rightarrow$ form of Sp_{2n} attached to $(A, *)$

is $\quad R \mapsto G(R) = \{a \in R \otimes_k A \mid a^*a = 1\}$

where $(Z(A) = k)$

A: central simple $/k$ with involution of first kind

& $(A, *)$ becomes isomorphic to $(M_{2n}, *)$

over $\bar{k}$

Forms of $\mathrm{spin}(\phi)$

(V, ϕ) nondegenerate quadratic space

over k with largest possible Witt index.

Forms of $\mathrm{spin}(\phi)$ are double covers of

forms of $SO(\phi)$

M: matrix of ϕ relative to V

$(M_n(k), *)$ $\quad X^* = MX^tM^{-1}$

$$X^*X = I \Leftrightarrow MX^tM^{-1} \cdot X = I$$

$$\Leftrightarrow MX^tM^{-1} = X^{-1} \Leftrightarrow XMX^t = M$$

$$\mathrm{Aut}(M_n(k), *) = O(\phi)/\text{center}$$

$\Rightarrow$ forms of $SO(\phi) \leftrightarrow$ forms of $(M_{2n}, *)$

$(A, *)$

$$G(R) = \{ a \in R \otimes_k A \mid a^* a = 1 \}$$

A central simple $/ k$ with involution of first kind and over $\bar{k}$ becomes isomorphic to $(M_n, *)$ $\quad X^* = M X^t M^{-1}$

M symmetric.

<u>Prop</u> If a central simple alg A over k admits an involution of first kind, then

$$A \otimes_k A \simeq M_{n^2}(k) \qquad n^2 = [A:k]$$

<u>pf</u> $A \otimes_k A^{opp} \to \mathrm{End}_{k\text{-lin}}(A)$ → regarded as k-vector space

<u>Thm</u> If k: alg. closed, finite field, $\mathbb{R}$, $\mathbb{Q}_p$ & finite exts, $\mathbb{Q}$ & finite exts.

the only central div. alg over k or a finite ext of k admitting an inv of first kind are quartenion algs & the field it self.

<u>Prop</u> If A is a quaternion alg of center L and has an involution $*$ of the second kind then

$A \cong B \otimes_k L$ where B is a k-algebra

k is the fixed field of $*$.

$*|_L$ = non-trivial k-automorphism.

$*|_B$ = standard involution. (conjugation).

§ Bilinear form & Hermitian forms.

discriminants.

D - divison algebra $x \mapsto \bar{x}$ involution

V = n-diml left vector space over D.

A sesquilinear form on V is

$$B : V \times V \to D$$

s.t $B(\beta x, \alpha y) = \beta B(x, y) \bar{\alpha}$ $\alpha, \beta \in D$, $x, y \in V$

B is Hermitian, if $B(x,y) = \overline{B(y,x)}$

skew Hermitian if $B(x,y) = -\overline{B(y,x)}$

Let $h(x) = B(x,x)$ $\leftarrow$ Hermitian / skew H. form.

Radical $(B) = \{x \in V \mid B(x,y) = 0 \quad \forall y \in V\}$

B nondegenerate if $\mathrm{Rad}(B) = 0$

Discriminant :

(V, Q) V/k Q quadratic form w/ matrix A

$\det A \in k^\times / k^{\times 2}$

(V, h) V/D h H./s.H form w/ matrix A

reduced norm of A $\in k^\times / k^{\times 2}$

Let A be a central simple k-alg w/ inv of first kind

i) $A \cong M_r(k)$

$\exists\ a \in A$ s.t $x^* = a^t x a^{-1}$ $\forall x \in A$

a either symmetric or skew-symmetric.

symmetric $\to$ type B, D

skew-symmetric $\to$ type C

ii) $A \cong M_s(C)$ C quaternion division algebra

$x^* = a \bar{x} a^{-1}$ $x \to \bar{x}$ standard inv.

$(x_{ij}) \to (\overline{x_{ji}})$

a Hermitian / skew Hermitian

Hermitian $\to$ C_n

skew Hermitian $\to$ D_n