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LLC for tori

Last time : F non-arch field

G reductive$/F$ and G splits over a tamely ramified extension

Consider a max torus S, elliptic, and maximally unramified

Given a regular charater $\theta : S(F) \to \mathbb{C}^\times$

$\rightsquigarrow$ regular supercuspidal π of $G(F)$

$(S, \theta) \rightsquigarrow$ Langlands parameter S

$\rightsquigarrow$ Langlands parameter for π

Let F be a p-adic field $\quad G_F = \mathrm{Gal}(\bar{F}/F)$

$W_F =$ Weil gp

① If M is a disrete G_F-mod, then

$H^i(G_F, M)$ is torsion for $i > 0$

② $-''-$, $H^i(G_F, M) = 0$ $\quad i > 2$ (Tate) ②

③ Think $H^*(G_F, M) = H^*(X, \mathcal{L})$

X = classifying space of G_F,

$\mathcal{L}$ l.s. associated to M.

Local class field theory

F'/F Galois finite ext $\quad \Gamma = Gal(F'/F)$

$$\Gamma = G_F/G_{F'} = W_F/W_{F'}$$

$$\begin{tikzcd} F^* \arrow[r, "\sim"] \arrow[d, shift right, "\text{ind}"'] & W_F^{ab} \arrow[d, shift right, "\text{transfer}"'] \\ F'^* \arrow[r, "\sim"] \arrow[u, shift right, "\text{norm}"'] & W_{F'}^{ab} \arrow[u, shift right, "\text{inclusion}"'] \end{tikzcd}$$

$$\begin{tikzcd} X_{F'} \arrow[d, "\pi"] & H_1(X_{F'}) = W_{F'}^{ab} \arrow[d] \\ X_F & H_1(X_F) = W_F^{ab} \arrow[u, bend left, "\text{trans}"] \end{tikzcd}$$

pull by 1-cycle

Rmk Add functoriality + require

$\pi_F \mapsto$ Frob $\quad \to$ Fix the LCFT.

Let T/F be a torus

G_F acts on $T_{\bar{F}}$ $\Rightarrow$ G_F acts on

$X^*(T) = \mathrm{Hom}(T_{\bar{F}}, \mathbb{G}_m)$

$\Rightarrow$ G_F action on $\check{T} = \mathbb{C}^\times \otimes_{\mathbb{Z}} X^*(T)$

<u>Exercise</u>

$$\left\{ \begin{tikzcd} & {}^L T \arrow[d] \\ W_F \arrow[ur] \arrow[r] & G_F \end{tikzcd} \right\} / \check{T} = H^1(W_F, \check{T})$$

<u>LLC</u>

$$\{ T(F) \to \mathbb{C}^\times \} \longleftrightarrow H^1(W_F, \check{T})$$

<u>Case 1</u> F'/F separable

$T = \mathrm{Res}_{F'/F}\, \mathbb{G}_m \qquad T(F) = (F')^\times$

$\check{T} = \mathrm{Ind}_{W_{F'}}^{W_F}(\mathbb{C}^*)$

$\mathrm{Hom}(T(F), \mathbb{C}^\times) = \mathrm{Hom}((F')^\times, \mathbb{C}^\times)$

$= \mathrm{Hom}(W_{F'}, \mathbb{C}^\times) = H^1(W_{F'}, \mathbb{C}^\times)$

$$= H^1(W_F, \mathrm{Ind}_{W_{F'}}^{W_F} \mathbb{C}^\times) = H^1(W_F, \check{T})$$

$$\begin{array}{cc} X_{F'} & H^1(X_{F'}, \mathcal{L}) \\ \downarrow \pi & \parallel \\ X_F & H^1(X_F, \pi_* \mathcal{L}) \end{array}$$

Let's consider two functors

$$\{\text{Tori}/F\} \longrightarrow \{\text{Abelian gps}\}$$

$$T \longmapsto \mathrm{Hom}(T(F), \mathbb{C}^\times)$$

$$T \longmapsto H^1(W_F, \check{T})$$

<u>Thm</u> There is a unique natural equivalence φ between these two functors s.t. it is the one in case 1 for induced tori

<u>pf</u> <u>uniqueness</u> Fix T/F choose F'/F Galois s.t T splits over F'

$$T' = \mathrm{Res}_{F'/F}(T \otimes_F F') \leftarrow T$$

a product of copies of $\mathrm{Res}_{F'/F} \mathbb{G}_m$.

$$\begin{tikzcd} \mathrm{Hom}(T(F), \mathbb{C}^\times) \arrow[r, "\varphi_T"] & H^1(W_F, \check{T}) \\ \mathrm{Hom}(T'(F), \mathbb{C}^\times) \arrow[u, twoheadrightarrow] \arrow[r, "\varphi_{T'}"'] & H^1(W_F, \check{T}') \arrow[u] \end{tikzcd}$$

$\hookrightarrow$ unique

<u>Existence</u> T/F F'/F Galois s.t $T_{F'}$ splits

$$1 \to W_{F'} \to W_F \to \Gamma \to 1$$

We have the corestriction map

$$H^1(W_{F'}, \check{T}) \to H^1(W_F, \check{T})$$

$$\begin{matrix} X_{F'} \\ \downarrow \\ X_F \end{matrix} \qquad H^1(X_{F'}, \pi^*\mathcal{L}) \cong H^1(X_F, \pi_*\pi^*\mathcal{L}) \underset{\text{natural}}{\overset{\text{trace}}{\rightleftarrows}} H^1(X_F, \mathcal{L})$$

$$\begin{tikzcd} \mathrm{Hom}(\overset{\substack{T(F') \\ \shortparallel}}{T_{F'}(F')}, \mathbb{C}^\times) \arrow[r, "\cong"] \arrow[d] & H^1(W_{F'}, \check{T}) \arrow[d] \\ \mathrm{Hom}(T(F), \mathbb{C}^\times) & H^1(W_F, \check{T}) \end{tikzcd}$$

proceed as step 1

$$\left.\begin{array}{l} \mathrm{Hom}(T(F), \mathbb{C}^\times) = \mathrm{Hom}(T(F'), \mathbb{C}^\times)_\Gamma \\ H^1(W_F, \check{T}) = H^1(W_{F'}, \check{T})_\Gamma \end{array}\right\} \Rightarrow \text{existence}$$

<u>pf</u> ① Replace $\mathbb{C}^\times$ by any conn. Lie gp

$\Rightarrow$ enough to prove for $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$

a) $\mathbb{R}$ $\quad \mathrm{Hom}(T(F'), \mathbb{R}) = \mathrm{Hom}(X_*(T) \otimes_{\mathbb{Z}} F'^*, \mathbb{R})$

$= \mathrm{Hom}(X_*(T), \mathbb{R})$

$\mathrm{Hom}(T(F), \mathbb{R}) = \mathrm{Hom}(X_*(T)^\Gamma, \mathbb{R}) = \mathrm{Hom}(X_*(T), \mathbb{R})_\Gamma$

b) $\mathbb{R}/\mathbb{Z}$ $\quad \mathrm{Hom}(T(F), \mathbb{R}/\mathbb{Z}) = T(F)^\times$ Pontryagian dual

<u>Claim</u> For any locally cpt abelian gp G with an action of a finite gp Γ we have $(G^\Gamma)^* = (G^*)_\Gamma \Rightarrow$ b)

② $H^1(W_{F'}, \check{T})_\Gamma \xrightarrow{\sim} H^1(W_F, \check{T})$

a) Replace $\mathbb{C}^\times$ by any divisible abelian gp E ⑦

$\hat{T} = X_{\mathbb{C}^\times} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times$ $\quad X_E = X^*(T) \otimes_{\mathbb{Z}} E$

Any divisible abelian gp is a direct sum of $\mathbb{Q}$'s and $\mathbb{Q}_p/\mathbb{Z}_p$'s

$\Rightarrow E = \mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$

b) $E = \mathbb{Q}$ $\quad H^1(W_{F'}, X_{\mathbb{Q}}) = H^1(\mathbb{Z}, X_{\mathbb{Q}}) = X_{\mathbb{Q}}$

Γ acts on $X_{\mathbb{Q}}$ the only coinv. comes from the trivial repn. i.e. from $X_{\mathbb{Q}}^{\Gamma}$

$H^1(W_F, X_{\mathbb{Q}}) = X_{\mathbb{Q}}^{\Gamma}$

c) $E = \mathbb{Q}/\mathbb{Z}$. $\quad H^1(W_F, X_{\mathbb{Q}/\mathbb{Z}}) = H^1(G_F, X_{\mathbb{Q}/\mathbb{Z}})$

$$1 \to X \to X_{\mathbb{Q}} \to X_{\mathbb{Q}/\mathbb{Z}} \to 1$$

$\Rightarrow H^1(G_F, X_{\mathbb{Q}/\mathbb{Z}}) \cong H^2(G_F, X)$ (same for F')

We are reduced to

$$H^2(G_F, X)_P \cong H^2(G_F, X)$$

We define a surjection

$$X' \to X \quad \text{s.t} \quad X' = \bigoplus \mathbb{Z}[P]$$

$$R = \ker(X' \to X) \qquad H^3(G_F, R) = 0$$

$$\begin{tikzcd}
H^2(G_F, R)_P \arrow[r] \arrow[d, "\text{surj}"] & H^2(G_F, X')_P \arrow[r] \arrow[d, "f_{X'}"] & H^2(G_{F'}, X)_P \arrow[r] \arrow[d, "f_X"] & 0 \\
H^2(G_F, R) \arrow[r] & H^2(G_F, X') \arrow[r] & H^2(G_F, X) \arrow[r] & 0
\end{tikzcd}$$

$$\left.\begin{array}{l} f_{X'} \text{ bijection} \Rightarrow f_X \text{ is surj} \\ f_R \text{ is surj} \end{array}\right\} \Rightarrow f_X \text{ iso}$$