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Quadratical reciprocity

$$\left(\frac{p}{q}\right) = \left(\frac{\pm q}{p}\right) \qquad p, q \text{ odd}$$

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$$

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & p \equiv \pm 1 \ (8) \\ -1 & p \equiv \pm 3 \ (8) \end{cases}$$

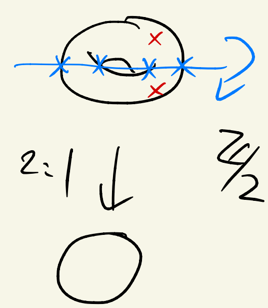
Artin's class field theory

L / K Galois, finite; $K / \mathbb{Q}$ finite; $L / \mathbb{Q}$ finite

assume G : abelian

$$\begin{array}{c} \operatorname{Spec} \mathcal{O}_L \\ \downarrow G \\ \operatorname{Spec} \mathcal{O}_K \end{array}$$

branched cover



$$\operatorname{max\,spec} \mathcal{O}_L / G \qquad \begin{array}{c} D_{\mathfrak{P}} = \operatorname{stab}_G \mathfrak{P} \\ \circlearrowright \\ \mathfrak{P} \quad \mathfrak{P}' \quad \mathfrak{P}'' \end{array}$$

$$\|$$

$$\operatorname{max\,Spec} \mathcal{O}_K \qquad \downarrow \qquad \mathfrak{p}$$

$$D_{\mathfrak{q}} \circlearrowright \mathcal{O}_L/\mathfrak{q} = k(\mathfrak{q})$$

$$\Big|$$

$$\mathcal{O}_K/\mathfrak{p} = k(\mathfrak{p})$$

$$D_{\mathfrak{q}} \twoheadrightarrow \mathrm{Gal}\left(k(\mathfrak{q})/k(\mathfrak{p})\right)$$

$$\mathrm{Frob}_{\mathfrak{q}} \in$$

(iso if unramified)

$$\bigoplus \mathfrak{p}^{\mathbb{Z}} \longrightarrow G \leftarrow \text{abelian}$$

$$\mathfrak{p} \longmapsto \mathrm{Frob}_{\mathfrak{q}}$$

// unram $\mathfrak{p}$

non-zero fract. ideals in K coprime to all ramified primes.

<u>Thm</u> $\exists$ conductor $\mathfrak{m}$ for L/K s.t.

if $\alpha \equiv 1 \ (\mathfrak{m})$ then $(\alpha) \mapsto 1 \in G$.

e.g. $\mathbb{Q}(\sqrt{\pm q})$ choose sign so $\equiv 1 \bmod 4$

$\quad\Big|$

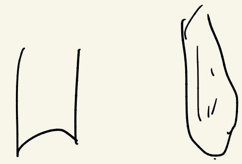
$\quad\mathbb{Q}$ can deduce Q.R. from Thm.

(primes talk to each other)

$$k^\times \backslash \mathbb{A}_k^\times / N_{L/k}\mathbb{A}_L^\times \xrightarrow{\ \simeq\ } G$$

(local Artin maps : all primes (nowdays))

$$\Gamma(N) \backslash \overline{\mathbb{H}} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\} := Y(N)$$



(modular curve of level N)

$$\Gamma(N) = \ker\left(SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})\right)$$

Conductor's

$\Updownarrow$

L's

$$SL_2(\mathbb{R}) \times SL_2(\mathbb{A}_\mathbb{Q}^f)$$

$$SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}_\mathbb{Q}) / SO(2) \times K_N$$

$$K_N = \ker\left(SL_2(\hat{\mathbb{Z}}) \to SL_2(\mathbb{Z}/N)\right)$$

(congruent completion of $\Gamma(N)$)

Take inverse limit

$$\Rightarrow \quad SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}_\mathbb{Q}) / SO(2)$$

$$\to \quad SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}_\mathbb{Q}) \circlearrowleft SL_2(\mathbb{A}_\mathbb{Q})$$

$=$ unit tangent bundle to $\varprojlim_N Y(N)$

$$\mathrm{Funs}\left(SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}_\mathbb{Q}), \mathbb{C}\right)$$ (regularities etc) $\leftarrow$ space of automorphic form for SL_2 over $\mathbb{Q}$ or $SL_2(\mathbb{A}_\mathbb{Q})$

- classical modular forms
- Siegal —
- Hilbert —
- Hil-Siegal —

$$= \mathrm{Ind}_{SL_2(\mathbb{Q})}^{SL_2(\mathbb{A}_\mathbb{Q})} \mathbb{1}$$

$$\Pi \circlearrowright SL_2(\mathbb{A}_\mathbb{Q})$$

$\Pi = \bigotimes_{v = \infty, 2, 3, \dots} \Pi_v$ irrep

Galois side

$$\pi_0\left(K \backslash \mathbb{A}_K^\times \right) \cong G_K^{ab}$$

$$\begin{tikzcd} G_\mathbb{Q} \arrow[r, "\text{cont. } \rho"] \arrow[d, two heads] & GL_2(\mathbb{C}) \ni \sigma_\mathfrak{q} \\ \mathrm{Gal}(L/\mathbb{Q}) \arrow[ur] & \end{tikzcd}$$

(factor through finite quot.)

$\mathrm{Gal}(L/\mathbb{Q}) \ni D_\mathfrak{q} = \langle \mathrm{Frob}_\mathfrak{q} \rangle$ if $\mathfrak{q}$ is unram.

$\sigma_\mathfrak{q}$ & $\sigma_{\mathfrak{q}'}$ conj. if $\mathfrak{q} \sim \mathfrak{q}'$ lie over same p

If p is unramified for ρ (i.e in L)

then char poly of $\sigma_{\mathfrak{q}}$ is indep of $\mathfrak{q}$

lying over p

$$(\text{unramified } p) \longmapsto \text{char polyns of } \sigma_{\mathfrak{q}}.$$

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free $\text{product}_p\ \mathbb{Z} \to GL_2(\mathbb{C})$

(Two ways of viewing $G_{\mathbb{Q}}$ as profinite gps)

($\mathbb{Z}^N$ is self-dual)

$$\rho: \langle \text{Frob}_p \rangle \to G(\mathbb{C})^{s.s}$$

$$\text{Frob}_p \longmapsto \sigma_p \in G(\mathbb{C}) \quad \text{up to conj.}$$

(assume σ_p s.s)

e.g. $G = GL_n$, σ_p is diagonalizable.

σ_p can be conjugated to lie in $T(\mathbb{C})$.

really, $\sigma_p \in (T/W)(\mathbb{C})$

$$G_{\mathbb{Q}_p}^{\text{tame}} \cong \mathbb{Z} \ltimes \mathbb{Z}[\tfrac{1}{p}]$$

$G_{\mathbb{Q}_p}^{\text{tame}} \twoheadleftarrow G_{\mathbb{Q}_p} \supset$ pro-p-gp

$\mathbb{Z} \ni$ Frob acts by mult by $\frac{1}{p}$

Unramified rep of $GL_n(\mathbb{Q}_p)$

π a smooth irrep of $GL_n(\mathbb{Q}_p)$ is unramified if $\pi^{GL_n(\mathbb{Z}_p)} \neq 0$

(spherical)

$\left({}^{SO_3}/_{SO_2} \right)$ spherical harmonics.

$$\mathbb{P}^1(\mathbb{Q}_p) = \frac{GL_2(\mathbb{Q}_p)}{B(\mathbb{Q}_p)} = \frac{GL_2(\mathbb{Z}_p)}{B(\mathbb{Z}_p)}$$

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} \mathbb{1} = \mathrm{Funs}(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{C})$$

$$= \mathrm{Ind}_{B(\mathbb{Z}_p)}^{GL_2(\mathbb{Z}_p)} \mathbb{1} \supseteq \mathbb{1}$$

$$\begin{pmatrix} x & \\ & x \end{pmatrix} \downarrow \begin{pmatrix} p^{\mathbb{Z}} & 0 \\ 0 & p^{\mathbb{Z}} \end{pmatrix} \qquad T(\mathbb{Q}_p) \downarrow {}^{T(\mathbb{Q}_p)}/_{T(\mathbb{Z}_p)}$$

$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} \chi$ if $\chi : B(\mathbb{Q}_p) \to T(\mathbb{Q}_p) \to \mathbb{C}^\times$

which is trivial on $B(\mathbb{Z}_p)$.

Irr. unless χ factors through the determinant.

<u>Thm</u> unram. reps are

(i) $\mathrm{Ind}_B^{GL_2} \chi$ (for unram. χ not factoring through det)

(ii) $\chi \circ \det$ (for $\chi : \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times / \mathbb{Z}_p^\times \to \mathbb{C}^\times$)

<u>pf</u> Hecke theory.

$$v \in \Pi^{GL_2(\mathbb{Z}_p)}$$

$$g \cdot v \in \Pi^{g\, GL_2(\mathbb{Z}_p)\, g^{-1} \cap GL_2(\mathbb{Z}_p)}$$

$$\sum_{h \in [GL_2(\mathbb{Z}_p) : g\, GL_2(\mathbb{Z}_p)\, g^{-1} \cap GL_2(\mathbb{Z}_p)]} h\, g\, v$$

$$\mathcal{H}\left(G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \right) \xrightarrow{\sim} \mathcal{H}\left(T(\mathbb{Q}_p) / T(\mathbb{Z}_p) \right)^{W}$$

$SL_2(\mathbb{Q}_p)$

$SL_2(\mathbb{Z}_p)$ max-compact (not conj.)

$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$?