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Proofs of LLC & CFT are global

JL endoscopic transfer (Arthur's book)

<u>Q</u> How to take a local thing & embed it into a global one?

<u>Q</u> let G be red. gp $/k$ k global

Let σ be an irrep of $G(k_v)$ v place of k

Does there exist an automorphism rep. $\pi = \bigotimes_w \pi_w$ of $G(\mathbb{A}_k)$ with $\pi_v \simeq \sigma$?

$G = GL_1$ $\sigma : k_v^\times \to \mathbb{C}^\times$

$$\begin{matrix} k_v^\times & \\ \cap & \\ \mathbb{A}^\times/k^\times & \to \mathbb{C}^\times \end{matrix}$$

construct $\pi : \mathbb{A}^\times/k^\times \to \mathbb{C}^\times$

<u>Comment</u>: Answer is no as stated.

by cardinality arguments (countable : autom. uncountable : local)

Rescue the question by requiring that

σ be cuspidal (for v non-arch.)

$k \rightsquigarrow F$

$$L^2\left(G(F) \backslash G(\mathbb{A}_F), \omega\right) \supseteq L^2_{cusp}(-)$$

ω ↓ fixed central char.

$\varphi \in L^2\left(G(F) \backslash G(\mathbb{A}_F)\right)$ is cuspidal if

$$\int_{U(F) \backslash U(\mathbb{A})} \varphi(ug)\, du = 0 \qquad \forall\, U \text{ unip radical of parabolics.}$$

$L^2_{cusp}(-)$ ← decomposes discretely into

$\oplus$ aut repn π with finite mult.

<u>Main tool</u> trace formula

<u>e.g.</u> $f \in \mathcal{S}(\mathbb{R})$ $\displaystyle\sum_{n \in \mathbb{Z}} \hat{f}(n) = \sum_{n \in \mathbb{Z}} f(n)$

(Poisson summ.) $= \operatorname{tr}\left(f, L^2(\mathbb{R}/\mathbb{Z})\right)$

In general study action of

$f \in C_c^\infty(G)$ on $L^2(\Gamma \backslash G)$ & compute trace

if $\varphi \in L^2(\Gamma \backslash G)$ $(f * \varphi)(x) = \int_G \varphi(xy) f(y) dy$

Much easier if $\Gamma \backslash G$ is compact.

<u>Interesting examples</u> $G = \underline{G}(\mathbb{A})$ $\Gamma = \underline{G}(F)$

$\underline{G}$ ss $\quad$ G red, need to consider fixed action of $Z(\mathbb{A})$

$\hookrightarrow$ center

$\Gamma \backslash G \iff G$ anisotropic (no split torus)

in general, finite volume, non-cpt.

(Assume no center)

$f \in C_c^\infty(G)$ $\quad$ $\varphi \in L^2(\Gamma \backslash G)$

$$(f * \varphi)(x) = \int_G \varphi(y) f(x^{-1}y)\, dy$$

$$= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \varphi(\gamma y)\, f(x^{-1}\gamma y)\, dy$$

$$= \int_{\Gamma\backslash G} \Big(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y) \Big)\, \varphi(y)\, dy$$

Let $k(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)$ cts fn on

$\Gamma\backslash G \times \Gamma\backslash G$

if $\Gamma\backslash G$ cpt $k(x,y) \in L^2(\Gamma\backslash G \times \Gamma\backslash G)$

$\Rightarrow f$ acts by a Hilbert - Schmidt operator.

$$\operatorname{tr} f = \int_{\Gamma\backslash G} k(x,x)\, dx$$

assuming no analysis issues.

$$= \int_{\Gamma\backslash G} \sum_{\substack{\gamma\in \text{conj}\\ \text{classes}\\ \text{of } \Gamma}} \sum_{\delta\in \Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\,\delta x)\, dx$$

$$= \sum_{\substack{\gamma\in \text{conj}\\ \text{classes of } \Gamma}} \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)\, dx$$

$$= \sum_{\substack{\gamma \in \text{conj} \\ \text{classes} \\ \text{of } \Gamma}} \mathrm{vol}\left(\Gamma_\gamma \backslash G_\gamma\right) \underbrace{\int_{G_\gamma \backslash G} f(x^{-1}\gamma x)\, dx}_{\Downarrow \text{ orbital integral}}$$

(if G is unimodular)

Geometric side of trace formula $\leftarrow$ ($\Gamma \backslash G$ cpt) $\rightarrow$ spectral side

$$L^2(\Gamma \backslash G) \simeq \bigoplus_{\substack{\pi \\ \text{unitary} \\ \text{irrep of } G}} m_\pi \pi \qquad m_\pi \in \mathbb{N}$$

$$\mathrm{tr}(f) = \sum_\pi m_\pi \, \mathrm{tr}(f, \pi)$$

$G = G(\mathbb{A}) \qquad \Gamma = G(F)$

$f \in C_c^\infty(G(\mathbb{A}))$ choose $f = \prod f_v$

$f_v \in C_c^\infty(G(F_v))$

$$\int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(x^{-1}\gamma x)\, dx = \prod_v \int_{G_\gamma(F_v) \backslash G(F_v)} f_v(x_v^{-1}\gamma_v x_v)\, dx_v$$

Problems that happen when $\Gamma \backslash G$ is not compact ⑥

1) $L^2(\Gamma\backslash G)$ does not have discrete spectrum.

Fix: think about $L^2_{cusp}(\Gamma\backslash G)$

Lemma $G = G(\mathbb{A})$ $\Gamma = G(F)$

Let $f = \prod f_v \in C_c^\infty(G(\mathbb{A}))$

Assume $\exists$ w s.t f_w is a matrix coeff of a cuspidal irrep of $G(F_w)$ (non-arch.)

Then f maps $L^2(G(\mathbb{F})\backslash G(\mathbb{A}))$ into $L^2_{cusp}(G(\mathbb{F})\backslash G(\mathbb{A}))$

Cor $tr(f, L^2) = tr(f, L^2_{cusp})$

Pf $\varphi \in L^2$

$$\int_{U(\mathbb{F})\backslash U(\mathbb{A})} (f * \varphi)(ug)\, du$$

$$= \int_{U(\mathbb{F})\backslash U(\mathbb{A})} \int_{G(\mathbb{A})} f(g^{-1}uh)\, \varphi(h)\, dh\, du$$

$$= \int_{U(F)\backslash U(\mathbb{A})} \int_{U(F)\backslash G(\mathbb{A})} \sum_{\gamma \in U(F)} f(g^{-1}\underbrace{u^{-1}\gamma}_{\text{absorb}} h)\, \underset{\varphi(h)}{\underset{\downarrow}{\varphi(\gamma h)}}\, dh\, du$$

$$= \int_{U(F)\backslash G(\mathbb{A})} \left(\int_{U(\mathbb{A})} f(g^{-1}uh)\, du \right) \varphi(h)\, dh$$

Inner integral $= \prod_v \int_{U(F_v)} f_v(g_v^{-1} u_v h_v)\, du_v$

<u>Claim</u> $\int_{U(F_w)} f_w(guh)\, du = 0$

The matrix coeff is $f_w(g) = \langle \pi(g) x, y \rangle$

$f_w(guh) = \langle \pi(u) x', y' \rangle$

π is cuspidal $\Rightarrow$ Jacquet modules are zero.

$x' = \sum_{i=1}^{n} (\pi(u_i) x_i - x_i)$

<u>Problems with non-cpt quotient</u> think: $\mathrm{vol}(F^\times \backslash \mathbb{A}^\times) = \infty$

- γ non-elliptic s.s $\mathrm{vol}\left(\frac{G_\gamma}{P_\gamma}\right) = \mathrm{vol}\left(\frac{G_\gamma(\mathbb{A})}{G_\gamma(F)}\right) = \infty$

- γ = unipotent e.g. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ SL_2 / PGL_2

$$\int = \prod_p \int_{\text{local}} = \prod_{p \text{ prime}} \frac{p}{p-1} = \infty$$

Only elliptic conj. classes behave well on geometric side. $\to$ G_γ anisotropic

<u>Thm</u> (simple trace formula)

Let $f \in C_c^\infty(G(\mathbb{A}))$ be s.t.

- f_{w_1} is matrix coeff of cuspidal repn.
- f_{w_2} is supported on elliptic elts.

Then
$$\sum_{\pi \subseteq L^2_{cusp}\left(G(\mathbb{A})/G(F)\right)} m_\pi \operatorname{tr}(f, \pi) = \sum_{\substack{\gamma \\ \text{elliptic}}} \operatorname{vol}\left(G_\gamma(\mathbb{A})/G_\gamma(F)\right)$$

$$\int_{G(\mathbb{A})/G_\gamma(\mathbb{A})} f(g^{-1}\gamma g)\, dg$$

<u>Goal</u> Given σ cusp irrep of $G(F_w)$

Find $\pi = \otimes \pi_v$ s.t $\pi_w \cong \sigma$

Stradegy Pick $f = \prod' f_v$

- simple trace formula holds for f
- geometric side $\neq 0$

$\Rightarrow \exists\ \pi$ s.t $tr(f, \pi) \neq 0$

Since $\pi = \otimes \pi_v$ $\prod tr(f_v, \pi_v) \neq 0$

Would be great if 3) $tr(f_w, \pi) = \begin{cases} \neq 0 & \pi \cong \sigma \\ = 0 & \pi \not\cong \sigma \end{cases}$

can you always find this??

$\Rightarrow \pi_w \cong \sigma$ then happy.

There is a theorem of Kazhdan (Tim K) which says that you get 3) if you restrict to tempered repns (for p-adic gps)

For GL_n, local components of automorphic cusp repns are tempered. (uses multiplicity one)

[not true for GSp_4]

(with f_w matrix coeff of σ]

$\uparrow$

orbital integrals of f_w = compute char of σ

can find γ s.t. orbital integral $\neq 0$ at place w.

orbital integral for $\gamma \neq 0$

shrink supp(f) so that every other orbital integral is zero

$$\begin{array}{ccc} G & & G(/A) \supset \overset{\text{cpt}}{\mathrm{supp}(f)} \\ \downarrow & & \Xi \downarrow \quad \swarrow \\ G/\!\!/G = C & & C(F) \subset C(/A) \end{array}$$

orbital integral only depends on

$$f\big|_{\Xi^{-1}(C(F))}$$

can modify f to vanish on all but the one fiber you care about.