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G locally compact totally disconnected gp

e.g $GL_2(F)$ F: non arch. local.

All repns are smooth: V is smooth $\forall v \in V$

$\exists$ K open s.t $gv = v$ $\forall g \in K$.

Let $H \leq G$ closed subgp

(W, ρ) a repn of H $\rho: H \to GL(W)$

Defn Compact induction

$$\text{C-Ind } W = \{ f: G \to W \quad f(hg) = \rho(h) f(g)$$

$$\exists\ K \text{ open s.t } f(gk) = f(g) \quad \forall k \in K \quad g \in G$$

$$\text{supp} f \text{ is cpt in } H \backslash G \}$$

G - acts by right translations.

- This is the left adjoint to Restriction.
- Ind W : same except drop compact condition.

Ind W is right adjoint to Res.

☆ H parabolic $\to$ parabolic induction

H cpt modulo center $\rightsquigarrow$ cuspidals.

e.g. W = trivial repn K-open subgp

$$\underset{\substack{\uparrow \\ \text{infinite diml}}}{(c\text{-}\mathrm{Ind}_H^G W)^K} = \{ f : \underset{\substack{\downarrow \\ \text{often infinite}}}{\underline{H \backslash G / K}} \to \mathbb{C} \}$$

e.g. $G = GL_n(F)$

$H = K = GL_n(\mathcal{O}_F)$.

Def Repn of G V is admissible if $\forall$ cpt open subgp K

$\dim V^K < \infty$. (finiteness condition)

Theorem $G = \mathcal{G}(F)$ $\mathcal{G}$ red $/F$ F: nonarch. local.

Then every irreducible smooth repn is admissible.

Exercise $G = SL_2$ H = Iwahori $= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} a,b,d \in \mathcal{O}_F \\ c \in \mathfrak{m} \\ ad-bc=1 \end{array} \right\}$

$$\begin{tikzcd} \chi : H \arrow[r] \arrow[dr, "d"'] & \mathbb{C}^\times \\ & (\mathcal{O}_F/\mathfrak{m})^\times \arrow[u] \end{tikzcd}$$

$c\text{-}\mathrm{Ind}_H^G \chi$ is not admissible.

$$\begin{array}{c} G \\ \uparrow c\text{-ind} \\ \text{max cpt } G(\mathcal{O}_F) \\ \uparrow \mathrm{Ind} \\ \text{Iwahori} \end{array}$$

$k_F = \mathcal{O}_F/\mathfrak{m}$.

covers inducing principal series repn

$SL_2(k_F) \rightsquigarrow SL_2(\mathcal{O}_F) \rightsquigarrow G$

Repn of $GL_2(k_F)$

$\rightsquigarrow$ principal series : Ind_B^G (summands)

cuspidals : $\dim = q+1$ come from charaters of non-split torus.

$V^u = \{0\}$ $\quad u = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ $\quad \mathbb{F}_{q^2}^\times \hookrightarrow GL_2(\mathbb{F}_q)$

<u>Thm</u> Let W be a cuspidal rep of $GL_2(k_F)$

$\xrightarrow{\text{inflate}}$ consider W as a repn of $\underset{\substack{\shortparallel \\ K}}{GL_2(\mathcal{O}_F)} \to GL_2(k_F)$

For any extension of W of $Z.K$ $\quad Z = \text{center}(GL_2)$

$c\text{-}\mathrm{Ind}_{ZK}^G W$ is irreducible & cuspidal.

(depth zero)

<u>pf</u> Let $Y \subset c\text{-}\mathrm{Ind}_{ZK}^G W$ be a nonzero submod.

$0 \neq \mathrm{Hom}_G(Y, c\text{-}\mathrm{Ind}_{ZK}^G W) \leq \mathrm{Hom}_G(Y, \mathrm{Ind}_{ZK}^G W)$

$= \mathrm{Hom}_{ZK}(Y, W)$

K profinite + Z central

$\Rightarrow$ Y is semisimple over ZK

Y has a copy of W.

Are there any copies of W in $c\text{-Ind}_{ZK}^{G} W$?

yes: fns supported on ZK.

$\rightarrow$ This copy generates $c\text{-Ind}_{ZK}^{G} W$.

<u>Goal</u> show there is only one copy of W in (as ZK-repn)

$c\text{-Ind}_{ZK}^{G} W$.

<u>This happens:</u>

$$\mathrm{Hom}_{ZK}(W, c\text{-Ind}_{ZK}^{G} W) \cong \mathrm{End}(c\text{-Ind}_{ZK}^{G} W)$$

(if c-ind irr, Schur's lemma says End is 1-diml)

$$\| $$

$$\{ f : G \to W \mid f(k_1 g k_2) = \rho(k_1) f(g) \rho(k_2) \quad k_1, k_2 \in KZ \}$$

A double coset KgK supports such a nonzero function

$\Longleftrightarrow$ $\mathrm{Hom}_{K^g \cap K}(W^g, W) \neq 0$

Def $g \in G$ intertwines W if

$$\mathrm{Hom}_{K^g \cap K}(W^g, W) \neq 0$$

Thm G-red. if (g intertwines $W \Leftrightarrow g \in KZ$)

then $c\text{-}\mathrm{Ind}_{ZK}^{G} W$ is irr. & cuspidal

Find 1 matrix coeff. of $c\text{-}\mathrm{Ind}\, W$ with cpt support mod center

$\xRightarrow{\text{irr (thm)}}$ all matrix coeff have compact support

$\xLeftrightarrow{\text{Thm}}$ cuspidal.

GL_2 $\quad W$ cusp repn of KZ $\quad K = GL_2(\mathcal{O}_F)$

$$_{KZ}\backslash G / KZ \cong \mathbb{N}$$

ϖ uniform. $\quad \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} \leftarrow n$

$g = \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}$ $\quad n \geq 1.$ $\quad K^g \cap K \supseteq U(\mathcal{O}_F) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

When restricted to $U(\mathcal{O}_F)$, W^g is trivial

W has no $U(\mathcal{O}_F)$-invariants as it is cuspidal.

This constructs depth zero repns.

groups in general (higher depth)

$E^\times K_N$ $\quad$ E/F quadratic

K_N = congruent subgp.