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A global field E is a finite extension of $\mathbb{Q}$ or the function field $\mathbb{F}_q(X)$ of a smooth proj. curve $/\mathbb{F}_q$

By a local field F we mean a finite ext of $\mathbb{Q}_p$, $\mathbb{F}_q((t))$, $\mathbb{R}$, $\mathbb{C}$

These are locally compact fields associated to absolute values. They are complete.

We write $\mathcal{O}_F$ = integers in F ($\mathbb{Z}_p$, $\mathbb{F}_q[[t]]$) in the non-archimedean case

Non-archimedean local fields can be thought of as a punctured disc.

Let G be a reductive alg gp $/F$. Then $G(F)$ is a locally cpt top. gp. It has a Haar measure.

If F is non-arch. then $G(F)$ has a cpct open subgp $G(\mathcal{O}_F)$. It has a nbhd basis

Consisting of cpt open subgps $\Rightarrow G(F)$ is totally disconnected. (no differential calc.)

If F arch $\Rightarrow G(F)$ is a Lie gp.

A complex repn of $G(F)$ is a top. v. space $V/\mathbb{C}$ (locally convex, Hausdorff, complete) together with a homomorphism

$$\pi: G(F) \to \mathrm{Aut}_{\mathbb{C}}(V) \quad \text{s.t}$$

$G(F) \times V \to V$ is continuous.

For F non-arch, for smooth repns, we put discrete top on $\mathbb{C}$. (thus on V)

non-arch: We call a smooth repn admissible if for all cpt open K, V^K is finite diml

arch: — " — if for a maximal cpt $K < G(F)$ $V|_K$ has each irr. rep of K appearing finitely many times.

Basic non-arch : Ngo's course notes
Bernstein lectures
Cartier (Corvalis 1977)

<u>Goal</u> Understand irreducible admissible repns of $G(F)$.

We call a repn cuspidal if it does not occur as compos. factor in parabolic induction from a proper Levi.

Non-arch : cuspidals are supercuspidal or compact cuspidal, i.e., their matrix coeff. have compact support (mod center)

The groups $G(F)$ are of type I. ($\mathbb{C}^*$-alg)

Thus they have a plancherel thm.

$$L^2(G) = \int_{\hat{G}_u} \pi \, d\mu$$

Irr. unitary repns → $\hat{G}_u$

(spectral theorem) for operators on Hibert space

If E is number field

G redutive $/E$ $\rightsquigarrow G(\mathbb{A}_E)$ locally cpt

$$L^2(G(E)\backslash G(\mathbb{A}_E)) = \int_{\widehat{G(\mathbb{A}_E)}_u} \pi \, d\mu \quad \text{(Langlands program)}$$

For p-adic groups, one construts supercuspidal repns by compact indution from repns of cpt open subgps (Y_u) For $p \nmid |W|$, these are all (Fintzen)

We are assuming G splits over a tamely ramified extension.

<u>To understand</u>

1) Basics : repn theory of totally discon. gps (e.g. Ngo's lectures)

2) What are all redutive gps $/F$?
What is their struture

} Bruhat-Tits

(Lisa Carbone : rank 1)

iii) super cuspidal repn?

Kaletha constructs repns associated to charaters of tori (for real gps, this is how one thinks about irr. adm. repns)

iv) Relationship to arithmetic (LLC)

Kaletha can do this for his repns.

v) character theory.

Galois gps

F non-arch. $F \supset \mathcal{O}_F \supset \underset{(\pi_F)}{\overset{}{\mathfrak{m}_F}} \supset \mathfrak{m}_F^2 \supset \cdots$

L/F Galois $G = \mathrm{Gal}(L/F)$

$$G_i = \{ s \in G \mid s|_{\mathcal{O}_L/\mathfrak{m}_L^{i+1}} = 1 \}$$

$$\underset{\substack{\parallel \\ G}}{G_{-1}} \supset \underset{\substack{\parallel \\ I}}{G_0} \supset G_1 \supset \cdots$$

$$\mathcal{O}_L^* \supset U_L^0 \supset U_L^i = 1 + \mathfrak{m}_L^{i+1}$$

Not hard to see

$$\mathrm{gr}\, G_0 \hookrightarrow \mathrm{gr}\, \mathcal{O}_L^* \Rightarrow I = G_0 \text{ is solvable}$$

$G/I = \mathrm{Gal}(k_L/k_F)$ k = residue fields

$\Rightarrow$ G solvable.

$$\mathcal{O}_L^\times / U_L^1 = k_L^* \qquad U_L^i / U_L^{i+1} = k_L \qquad i \geq 1.$$

Rmk 1) The lower numbering does not behave well in quotients. So have to move to upper numbering (it is by real numbers)

Now get a filtration of $\mathrm{Gal}(\bar{F}/F)$

2) The upper numbering is consistent with the Moy-Prasad filtration. In particular it is compatible with class field theory.

The extension of F corresp. to I is F^{un},

$$\mathrm{Gal}(F^{un}/F) = \mathrm{Gal}(\bar{k}_F/k_F)$$

The F^{un} we get by adjoining to F all roots of unity coprime to $|k_F|$

$$\begin{array}{l} F^{tr} \\ \quad | \leftarrow \text{ adjoin all } \sqrt[p^n-1]{p} \\ F^{un} \\ \quad | \\ F \end{array} \qquad F = \mathbb{Q}_p$$

$I/G_1 \cong \overline{\mathbb{F}}_p^*$ ↺ the action by $\mathrm{Gal}(\bar{k}_F/k_F)$ is given by $Fr: x \to x^p$

$\mathrm{Gal}(F^{tr}/F) = G/G_1$

$$1 \to I/G_1 \to G/G_1 \to \hat{\mathbb{Z}} \to 1$$

$$G/G_1 = \prod_{\ell \neq p} \mathbb{Z}_\ell \rtimes \hat{\mathbb{Z}}$$

$$\prod_{\ell \neq p} \mathbb{Z}_\ell = \varprojlim_{L} k_L^\times$$

Groups Let G/F redutive then G_F is a redutive. These are classified by root datum $T \subset G_F$ maximal torus $(X^*(T), \Phi, X_*(T), \Phi^\vee)$

We also have G^{split}/F the split gp. we use it as a base point.

All F-forms of $G_{\bar{F}}$ are parametrized by

$$H^1(\Gamma, \mathrm{Aut}(G_{\bar{F}})) \quad \Gamma = \mathrm{Gal}(\bar{F}/F).$$

$$1 \to G_{\bar{F}}^{ad} \to \mathrm{Aut}(G_{\bar{F}}) \to \mathrm{Out}(G_{\bar{F}}) \to 1$$

splits by fixing a pinning.

$$H^1(\Gamma, G_{\bar{F}}^{ad}) \to H^1(\Gamma, \mathrm{Aut}(G_{\bar{F}})) \to H^1(\Gamma, \mathrm{Out}(G_{\bar{F}})) \to 1$$

<u>Fact</u> For non-arch, $H^1(\Gamma, G_{\bar{F}}^{sc}) = 0$ (If F finite, ok for all G)

If G s.s, $H^1(\Gamma, G_{\bar{F}}^{ad}) \cong H^2(\Gamma, Z(G_{\bar{F}}^{sc}))$

(Does anyone want to explain?)

$$\mathrm{Out}(G_{\bar{F}}) = \mathrm{Out}(\check{G}_{\mathbb{C}}) \qquad {}^LG = \check{G}_{\mathbb{C}} \rtimes \Gamma.$$

(Maybe $\check{G}_{\mathbb{C}}$ is not exactly right, geom. Satake does not naturally give this)

<u>Weil gp</u>

F non-arch $\qquad 1 \to I \to W_F \to \mathbb{Z} \hookrightarrow \hat{\mathbb{Z}} \to 1$

$\mathbb{C}$ $\qquad W_{\mathbb{C}} = \mathbb{C}^\times$

$\mathbb{R}$ $\quad 1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to \Gamma \to 1$ $\quad \Gamma = \mathbb{Z}/2\mathbb{Z}$ acts by cpx conj.

unique non-trivial ext.

↑

fundamental class of class field theory $H^2(\Gamma, \mathbb{C}^\times)$

Weil gp for global # fields is gigantic & not easy to constr.

(L)LC :

$$\Phi : \left\{ \begin{array}{ccc} & & {}^L G \\ & \nearrow^{\varphi} & \downarrow \\ W_F & \longrightarrow & \Gamma \end{array} \right\} \Big/ \check{G}_{\mathbb{C}}$$

" $\longleftrightarrow$ " {irr. adm. repn of inner forms of $G(F)$}

Modifications

① $W_F \rightsquigarrow \mathcal{L}_F$ Langlands gp

For local fields, the Weil-Deligne gp

② Depends on what kind of repn you want.

Arthur gp $\rightsquigarrow$ All automorphic repns occuring in the discrete spectrum L^2

③ The parameters give us L-packets. Perhaps should view $Z = \Phi/\check{G}$ as a space and the parameters should be (perverse) sheaves on this space.

Weil-Deligne gp

$W_F' = \mathbb{G}_a \rtimes W_F$ $\qquad wxw^{-1} = \|w\|x$.

LLC $\quad \Phi = \{ W_F' \xrightarrow{\varphi} {}^LG \}$
i) continuous
ii) φ maps s.s to s.s.
iii) φ maps $\mathbb{G}_a$ to unip elts.

Thm $\quad \{$cpx repns of $W_F'\}$

$$\updownarrow$$

$\{ \ell$-adic repns of $W_F\}$

$\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ $\qquad q = p^n$

$\mathbb{Q}_p^{tr}$ (tamely ramified)

$$\begin{array}{c} \mathbb{Q}_p^{tr} \\ | \\ \mathbb{Q}_p^{ur} \\ | \\ \mathbb{Q}_p \end{array}$$

$$\left.\begin{array}{c} \mathbb{Q}_p(\zeta_{q-1})(\sqrt[q-1]{p}) = E \\ \Big| \ \mathbb{F}_q^* \\ \mathbb{Q}_p(\zeta_{q-1}) \\ \Big\downarrow \ \mathbb{Z}/n\mathbb{Z} \\ \mathbb{Q}_p \end{array}\right\} \text{Kummer}$$

Frob acts by pth power

$$\mathrm{Gal}(E/\mathbb{Q}_p) = \langle \sigma, \tau \rangle \Big/ \langle \sigma^n = 1,\ \tau^{q-1} = 1,\ \sigma\tau\sigma^{-1} = \tau^p \rangle$$

$\tau \in \mathbb{F}_q^*$

$$\rightsquigarrow$$

$$\langle \sigma, \tau \rangle \Big/ \langle \sigma\tau\sigma^{-1} = \tau^p \rangle$$

$$\Big\downarrow \text{ profinite completion}$$

$$\prod_{\ell \neq p} \hat{\mathbb{Z}}_\ell(1) \cong \begin{pmatrix} \text{All roots} \\ \text{of unity} \\ \text{prime to } p \end{pmatrix}$$

$$1 \to \begin{pmatrix} \text{All roots} \\ \text{of unity} \\ \text{prime to } p \end{pmatrix} \to \mathrm{Gal}(\mathbb{Q}_p^{tr}/\mathbb{Q}_p) \to \hat{\mathbb{Z}} \to 1$$

$$\mathrm{Gal}(\mathbb{Q}_p^{tr}/\mathbb{Q}_p) = G/G_1$$

Rmk: Maximal abelian ext of $\mathbb{Q}_p$

add $\sqrt[p^\infty]{p}$.

LLC for tori

class field theory: $H_T^i(Gal(E/F), \mathbb{Z})$ (Tate gp)

$$\downarrow \wr$$

cup with $\alpha_{E/F} \in H^2(Gal(E/F), E^*)$ $\quad H_T^{i+2}(Gal(E/F), E^*)$

E/F finite extension. F local field

$$\begin{array}{ccc} H_T^{-2}(Gal(E/F), \mathbb{Z}) & \xrightarrow{\cong} & H_T^0(Gal(E/F), E^*) \\ \| & & \| \\ H_1(Gal(E/F), \mathbb{Z}) & & F^*/Norm_{E/F}(E^*) \\ \| & & \\ Gal(E/F)^{ab} & \xrightarrow{\cong} & \end{array}$$

class field theory

$$\left.\begin{array}{ccc} W_F^{ab} & \xrightarrow{\cong} & F^* \\ \Psi & & \\ Frob & \longmapsto & [\pi_F] \end{array}\right\}$$

(passing to limit)

This is compatible with the upper index

filtration of W_F^{ab} & the $U^{\cdot}$ filtration of F^* ⑬

Let T/F be a torus

$$X^*(T) = X^*(T_{\bar F}) \circlearrowleft Gal(\bar F/F)$$

$$\check T(\mathbb{C}) = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^* \circlearrowleft Gal(\bar F/F)$$

LLC

$$\{W_F \to {}^L T\}/\check T_{\mathbb{C}} \cong \{T(F) \to \mathbb{C}^\times\}$$

$\wr|$ exercise

$$H^1(W_F, \check T_{\mathbb{C}})$$

Let F'/F separable set $T = Res_{F'/F}(\mathbb{G}_m)$

$$T(F) = (F')^* \qquad \check T = Ind_{W'}^{W} \mathbb{C}^*$$

$$H^1(W, \check T_{\mathbb{C}}) = H^1(W, Ind_{W'}^{W}(\mathbb{C}^\times))$$

$$= H^1(W', \mathbb{C}^*) = Hom(W', \mathbb{C}^*) = Hom(F'^*, \mathbb{C}^*)$$

$$= Hom(T(F), \mathbb{C}^\times)$$

If you insist of a functorial LLC compatible with above, get a unique LLC.

(upper indexing $\longleftrightarrow$ Moy-Prasad filtration)

Yu's ottawa lecture.