Oct 15, 2019

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Bruhat-Tits building

Buildings What is a building? Philosophy

(a) The building is a geometric object on which G acts.

(b) The building is a simplicial complex and/or a metric space.

JK Yu - ottawa   MacDonald's spherical functions on p-adic gps ~1972

(c) The building is G/B

One definition (very useful, not perfect)

Let G be a gp (any gp) B a subgp

Let W be a set of representatives of B-double cosets $G = \bigsqcup_{w \in W} BwB$

The building is $\mathcal{B} = G/B$ with

$\delta : \mathcal{B} \times \mathcal{B} \to W$ given by

$\delta(g_1 B, g_2 B) = w$ if $B g_1^{-1} g_2 B = BwB$.

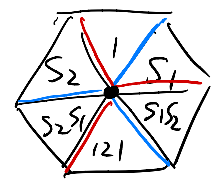
"Let me draw a picture"

Example $G = W_{fin} = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$

acts on "coxeter complex" a hexagon

"buiding relation"

$B = \{1\}$ is $s_1 s_2 s_1 = s_2 s_1 s_2$.



One triangle for each coset gB

One blue edge for each coset gP_1 $P_1 = \{1, s_1\}$

One red — gP_2 $P_2 = \{1, s_2\}$

One vertex for —"— gP $P = G$

Example $G = GL_3(\mathbb{F}_2)$ with $B = B(\mathbb{F}_2)$

Let $Y_1(c) = \begin{pmatrix} c & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $Y_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & -1 & 0 \end{pmatrix}$ $c \in \mathbb{F}_2$

$s_1 = y_1(0) \qquad s_2 = y_2(0)$

Then

$$G = B \sqcup Bs_1B \sqcup \cdots \sqcup Bs_1s_2s_1B$$

$$Bs_iB = \{y_i(c)B \mid c \in \mathbb{F}_2\}$$

$$Bs_1s_2B = \{y_1(c_1)\, y_2(c_2)\, B \mid c_1, c_2 \in \mathbb{F}_2\}$$

$$Bs_2s_1B = \{y_2(c_1)\, y_1(c_2)\, B \mid \text{—''—}\}$$

$$Bs_1s_2s_1B = \{y_1(c_1)\, y_2(c_2)\, y_1(c_3)\, B \mid c_1, c_2, c_3 \in \mathbb{F}_2\}$$

The building $\mathcal{B}$ has

One triangle for each coset gB

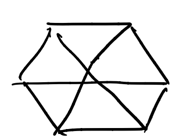
One blue edge —''— $gP_1 \quad P_1 = B \sqcup Bs_1B$

gP_2

G

"The building relation"

$$y_1(c_1)\, y_2(c_2)\, y_1(c_3) = y_2(c_3)\, y_1(c_1c_3 - c_2)\, y_2(c_1)$$

<u>Apartments</u> each apartment is isomorphic to the Coxeter complex of W_{fin} 

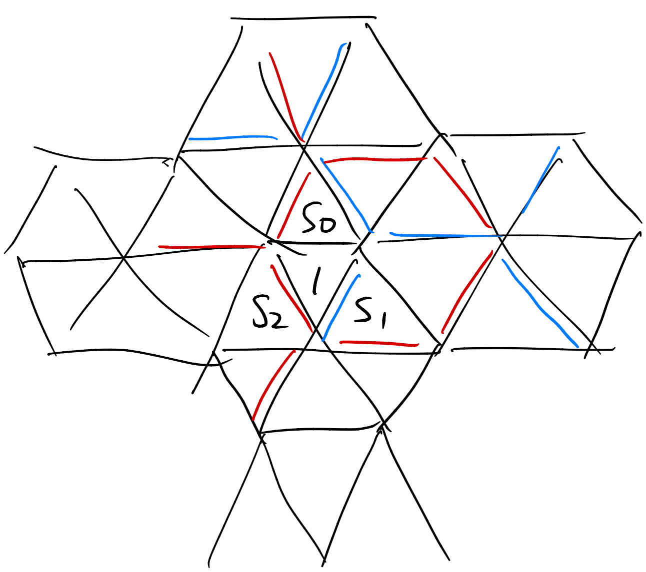
Fix $c_1, c_2, c_3 \in \mathbb{F}_2$

$$A_{c_1,c_2,c_3} = \begin{array}{cccc} & y_1(c_1)B & y_1(c_1)\, y_2(c_2) B & \\ B & & & y_1(c_1) y_2(c_2) y_3(c_3) B \\ & y_2(c_3) B & y_2(c_3)\, y_1(c_1c_3 - c_2) B & \end{array}$$

<u>Example</u> "The affine Coxeter cpx"

$$W_{af} = \langle s_0, s_1, s_2 \mid s_i^2 = 1 \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$i \in \mathbb{Z}/3\mathbb{Z} \rangle \qquad B = \{1\}$$



$P_1 = \{1, s_1\}$ blue

$P_2 = \{1, s_2\}$ red

$P_0 = \{1, s_0\}$ orange

<u>Example</u> "The $\widetilde{A}_2$ building"

$$G = GL_3(\mathbb{F}_2((\epsilon))) \qquad B = I = \{ g \in G \mid g(0) \text{ exists}$$
$$g(0) \in B(\mathbb{F}_2) \}$$

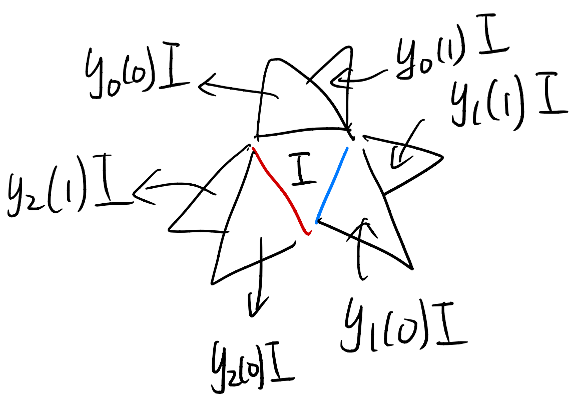
$$G = \bigsqcup_{w \in W_{af}} IwI \quad \text{where}$$

if $w = s_{i_1} \cdots s_{i_l}$ is reduced, then

$$IwI = \{ y_{i_1}(c_1) \cdots y_{i_l}(c_l) I \mid c_1, \cdots c_l \in \mathbb{F}_2 \}$$

where $y_1(c) = \begin{pmatrix} c & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad y_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & -1 & 0 \end{pmatrix}$

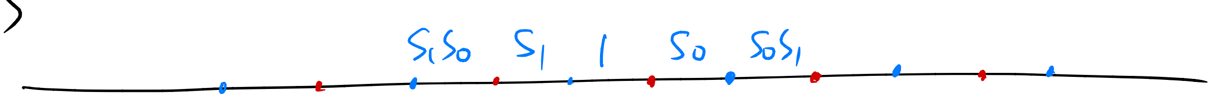
$$y_0(c) = \begin{pmatrix} c & 0 & -\epsilon^{-1} \\ 0 & 1 & 0 \\ \epsilon & 0 & 0 \end{pmatrix}$$



<u>Example</u> "The affine Coxeter complex" $\tilde{A}_1$

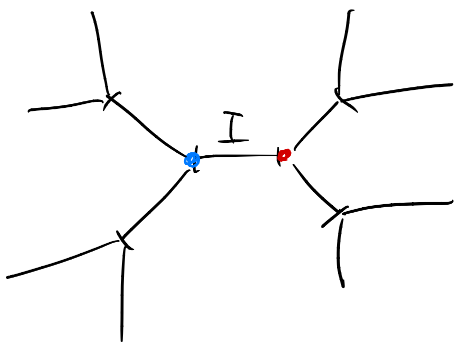
$W_{af} = \langle s_0, s_1 \mid s_i^2 = 1 \rangle$ infinite dihedral

$B = \{1\}$



$P_0 = \{1, s_0\}$ $\quad P_1 = \{1, s_1\}$

<u>Example</u> $G = GL_2(\mathbb{F}_2((t)))$



Regions in $\mathcal{B}$ correspond to useful subsets of G

Let $\mathcal{B}$ be the building

C a favorite chamber.

$\mathcal{A}$ a fixed apartment containing C

<u>Automorphism and stabilizers</u> (in the good case)

$G = \mathrm{Aut}(\mathcal{B})$ (want, but not precise)

$B = \{g \in G \mid gC = C\}$ "Borel subgp"

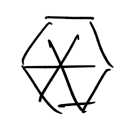
$N = \{ g \in G \mid g\mathfrak{h} = \mathfrak{h} \}$ "normaliser of $\mathfrak{h}$"

(B, N)-pairs

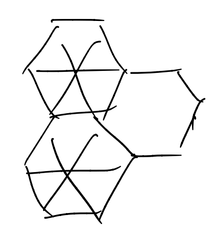
$H = \{ g \in G \mid g$ fixes $\mathfrak{h}$ pointwise$\}$ "Cartan subgp"

Groups like $GL_3(\mathbb{F}_2((\epsilon)))$ have two diff. buildings (which do talk to each other)

$\mathcal{B} = G/B = \{$"Borel subgps of $G\}$



$I = G/I = \{$"Iwahori subgps of $G\}$



By taking stabilizers

simplicies in $\mathcal{B}$ / I $\longleftrightarrow$ parabolics / parahorics in G

chambers in $\mathcal{B}$ / I $\longleftrightarrow$ minimal parabolics / parahorics in G

vertices in $\mathcal{B}$ / I $\longleftrightarrow$ maximal parabolics / parahorics in G

apartments in I $\longleftrightarrow$ maximal split tori in G

sectors in I $\longleftrightarrow$ parabolics in G

For $\alpha \in \mathring{R}$

the hyperplanes $\mathfrak{h}^\alpha \longleftrightarrow U_\alpha$ where

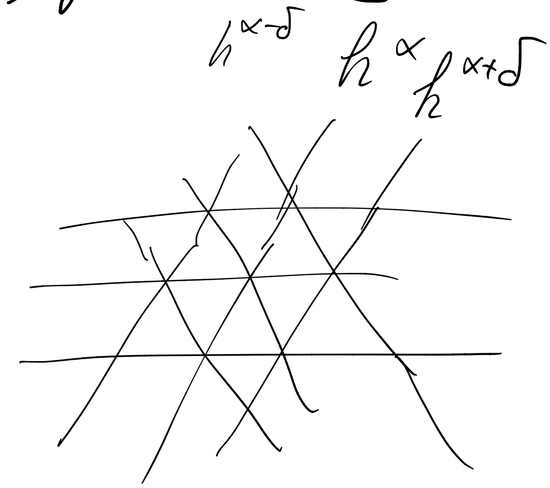
U_α is the filtered sequence

$$U_{\alpha-2\delta} \geq U_{\alpha-\delta} \geq U_\alpha \geq U_{\alpha+\delta} \geq \cdots$$

where $U_{\alpha+k\delta} = \{ x_\alpha(f) \mid f \in \epsilon^k \mathbb{F}_2[[\epsilon]] \}$

$$x_\alpha : \mathbb{G}_a \to G$$

The quotients in this sequence are "strips" in $\mathcal{B}$



$$\mathfrak{h} = \mathbb{R} \otimes_{\mathbb{Z}} X_*(T)$$

This is the source of Moy–Prasad filtrations.