Regular supercuspidal repns

F: non-archemedean local field

G: connected redutive gp defined over F

π: irreducible supercuspidal repn of $G(F)$ of depth zero.

Moy–Prasad '96 : $\exists$ a vertex $x \in \mathcal{B}^{red}(G,F)$

(reduced Bruhat–Tits building of $G(F)$)

s.t the restriction

$\pi|_{G(F)_{x,0}}$ contains the inflation to $G(F)_{x,0}$ of an irr. cuspidal repn

of $G(F)_{x,0:0+} := \dfrac{G(F)_{x,0}}{G(F)_{x,0+}}$

$G(F)_x = \mathrm{stab}_{G(F)}(x)$

$G(F)_{x,0}$ – parahoric subgp associated to x

$G(F)_{x,r}$ Moy–Prasad filtration gp $r \in \mathbb{R}_{\geq 0}$

Def π is regular (resp. extra regular) if

κ is a Deligne-Lusztig cuspidal repn

$\pm R_{S',\bar{\theta}}$ associated to an elliptic

maximal torus S' of G_x° and

a character $\bar{\theta} : S'(k_F) \to \mathbb{C}^\times$ that

is regular (resp. extra regular).

1) Bruhat-Tits : $x \in \mathcal{B}^{red}(G, F)$

$\rightarrow$ smooth connected $\mathcal{O}_F$-gp scheme

G_x° with $G_x^\circ(F) = G(F)$ $G_x^\circ(\mathcal{O}_F) = G(F)_{x,0}$

G_x° : redutive quotient of the special

fiber of G_x°

Then $\mathsf{G}_x^\circ(k_F) = G(F)_{x,0:0+}$

2) $\bar{\theta}$ regular if its stabilizer in

$\frac{N(S,G)(F)}{S(F)}$ is trivial. $\bar{\theta}$ is extra regular if its stabilizer in $\Omega(S,G)(F)$ is trivial $\quad (\Omega(S,G) = N(S,G)/S)$

Here S is a maximally unramified elliptic maximal torus $S < G$ s.t the redutive quotient of the special fiber of the conneted Nero model of S is $\mathcal{S}'$

[Maximally unramified elliptic maximal torus:

Fact $S \subset G$ max torus $\quad S' \subset S$ max. unramified subtorus

TFAE:

(i) S' is of maximal dim among the unramified subtori of G

(ii) S' not properly contained in an unramified subtorus of G

(iii) S is the centraliser of S' in G.

(4) $S \times F^u$ is a minimal Levi subgp of $G \times F^u$

(5) The action of I_F on $R(S, G)$ preserves a set of positive roots.

(<u>pf</u> $G \times F^u$ is quasi-split)

A max torus satisfying the above equivalence conditions called maximally unramified. }

<u>Kaletha '2019</u> Every regular depth-zero supercuspidal repn of $G(F)$ is of the form $\pi_{(S,\theta)}$ S: maximally unramified elliptic maximal torus

$\theta : S(F) \to \mathbb{C}^\times$ regular depth-zero character (i.e. $\theta|_{S(F)_0}$ equals the inflation of a regular $\bar{\theta}$)

[$S(F)_0 = S(F) \cap G(F)_{x_S,0}$

$x_S \in \mathcal{B}^{red}(G,F)$ is associated to S

as follows:

$S' \subset S$ becomes maximal split over

F^u, $A^{red}(S,F^u) \subset \mathcal{B}^{red}(G,F^u)$

$\swarrow$

Frob. inv (since S is defined over F)

contains a unique Frob-fixed pt x_S

(S is elliptic $\Rightarrow$)

If G is simply-connected or adjoint

then $S(F) = S(F)_0$

$$\frac{S'(F)_0}{S'(F)_{0+}} = \frac{S(F)_0}{S(F)_{0+}} = S'(k_F)$$

$S(F)_r = \{ s \in S(F)_0 \mid \forall\, \chi \in X^*(S),$

$\mathrm{ord}(\chi(s) - 1) \geq r \}$ $(r > 0)$]

Two repns $\pi_{(S_1,\theta_1)}$ and $\pi_{(S_2,\theta_2)}$ are isomorphic $\Longleftrightarrow$ $(S_1,\theta_1) \sim (S_2,\theta_2)$

$\uparrow$ $G(F)$-conjugate.

Definition of $\pi_{(S,\theta)}$

S: maximally unramified elliptic max torus $\subset G$

$\theta: S(F) \to \mathbb{C}^\times$ regular depth zero char

$\theta|_{S(F)_0}$ factors through $\bar{\theta}: S(F)_{0:0+}$

x_S: vertex associated to S

$\rightsquigarrow \kappa_{(S,\bar{\theta})} = \pm R_{S',\bar{\theta}}$: irr. cusp. DL repn of $G_x^\circ(k_F)$ inflate to $G(F)_{x,0}$

$N_{G(F)_x}(\kappa_{(S,\bar{\theta})}) = S(F)\cdot G(F)_{x,0}$

S unramified easy

otherwise more work

Extend $\kappa_{(S,\bar{\theta})}$ to $S(F)\cdot G(F)_{x,0}$

$\pi_{(S,\theta)} = \text{c-Ind}_{S(F)\cdot G(F)_{x,0}}^{G(F)} \kappa_{(S,\theta)}$ irr supercuspidal

Assume G splits over a tame extension of F. $\mathrm{char}(k_F) \neq 2$

Regular supercuspidal repns of positive depth

Hakim–Murnaghan:

$\{$(extra) regular supercuspidal repns$\}$

$\leftrightarrow \{$ G-equivalence classes of (extra) regular Yu-data$\}$

Let $((G^0 \subsetneq G^1 \subsetneq \cdots \subsetneq G^d), \pi_{-1}, (\phi_0, \phi_1, \cdots \phi_d))$ be a reduced generic cuspidal G-datum.

G^i: tame twisted Levi subgp of G

i.e. conn. red. gp defined over F & becomes a Levi subgp of G over a tame extension of F, $G^d = G$.

π_{-1}: depth-zero supercuspidal repn of $G^0(F)$

$\phi_i : G^i(F) \to \mathbb{C}^\times$ smooth char of

depth $r_i > 0$ G^{i+1}-generic when $i \neq d$

with extra conditions

J-k Yu '01 :

$\{$reduced generic cuspidal G-data$\}$

$\downarrow$

$\{$ isom. classes of irr. supercuspidal

repns of $G(F)\}$

Hakim - Murnaghan 08' fibers of the map

Def $((G^0 \subsetneq G^1 \subsetneq \cdots \subsetneq G^d), \pi_{-1}, (\phi_0, \cdots \phi_d))$

reduced generic cuspidal G-datum is

called

1) regular, if π_{-1} is a regular

depth zero supercuspidal repn of

$G^\circ(F)$

2) extra regular, if it is normalised (i.e., pull-back of ϕ_i to $G^i_{sc}(F)$ is trivial $\forall 0 \le i \le d$) and π_{-1} is an extra regular depth zero supercuspidal repn of $G^\circ(F)$.

<u>Def</u> Tame regular elliptic pairs

$S \subset G$ max torus $\quad \theta : S(F) \to \mathbb{C}^\times$

(S, θ) is called tame elliptic regular (resp. tame elliptic extra regular)

if

1) S is elliptic and split over a tame extension

2) the action of inertia on the root subsystem $R_{0+} = \{ \alpha \in R(S,G) \mid \theta(N_{E/F}(\alpha^\vee(E^\times_{0+})) = 1 \}$

preserves a set of positive roots where E/F is any tame Galois extension splitting S. (R_{0+} indep of choice of E/F) $E_0^\times = \mathcal{O}_E^\times$ $E_r^\times = 1 + \mathfrak{p}_E^{\lceil er \rceil}$ e ramification degree of E/F.

3) the character $\theta|_{S(F)_0}$ has trivial stabiliser for the action of

$$N(S, G^0)(F) \big/ S(F) \qquad (\text{resp. } \Omega(S, G^0)(F))$$

Where $G^0 \subset G$ is the redutive subgp with max torus S & root system R_{0+}.

2) $\Leftrightarrow$ S is a maximally unramified maximal torus of G^0

[Generalisation of admissible chars for GL_n]

Kaletha 2019' Assume p is not a bad prime for G & $p \nmid |\pi_1(G_{der})|$

This condition can be removed.

<u>Def</u> $p \nmid |\pi_1(G_{der})|$. A supercuspidal repn of $G(F)$ is called (extra) regular if it arises via Yu's construction from an (extra) regular (reduced generic cuspidal) Yu-datum.

$\{G(F)$ - conj classes of (extra) regular tame elliptic pairs$\}$

$\xleftrightarrow{1-1}$ $\{$(extra) regular supercuspidal repns$\}$

$$(S, \theta) \longmapsto \pi_{(S,\theta)}$$

$$((G^0 \subsetneq G^1 \subsetneq \cdots \subsetneq G^d), \pi_{-1}, (\phi_0, \cdots \phi_d))$$

classification of depth-zero (extra) regular supercuspidal

$\Rightarrow \quad \pi_{-1} = \pi(S, \phi_{-1})$

$S \subset G^0$: maximally unramified elliptic max torus.

$\phi_{-1} : S(F) \to \mathbb{C}^\times$ (extra) regular depth zero char

$\theta = \prod_{i=-1}^{d} \phi_i |_{S(F)}$

Regular supercuspidal L-packets

G - conn. reductive gp defined $/F$

quasi-split $/F$

split / a tame extension of F

${}^LG = \hat{G} \rtimes W_F$

p is not a bad prime for G

and $p \nmid |\pi_0(Z(G))|$

<u>Def</u> A strongly regular supercuspidal parameter is a discrete Langlands parameter

$\varphi : W_F \to {}^LG$ s.t

$\varphi(P_F) \subset$ torus of $\hat{G}$

$\mathrm{Cent}(\varphi(I_F), \hat{G})$ abelian

P_F: wild inertia

Def A regular supercuspidal parameter is a discrete Langlands parameter $\varphi: W_F \to {}^LG$ satisfying

i) $\varphi(P_F) \subset$ torus of $\hat{G}$. let $\hat{M} = \mathrm{Cent}(\varphi(P_F), \hat{G})^\circ$

ii) $C := \mathrm{Cent}(\varphi(I_F, \hat{G}))^\circ$ is a torus

Let $\hat{S}$: $\Gamma = \mathrm{Gal}(F^s/F)$-mod: underlying abelian gp $\hat{T} = \mathrm{Cent}(C, \hat{M})$

Γ action given by $\mathrm{Ad}(\varphi(-))$

iii) If $n \in N(\hat{T}, \hat{M})$ projects onto a non-trivial elt of $\Omega(\hat{S}, \hat{M})^\Gamma$, then $n \notin \mathrm{Cent}(\varphi(I_F), \hat{G})$

[strongly regular $\Rightarrow$ regular

Almost all regular strongly regular]