Lecture by Peter McNamara

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<u>Goal</u>: Beilinson - Bernstein localization.

$$\left\{ \begin{array}{c} \mathfrak{g}\text{-mod. with a fixed} \\ \text{central char} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{twisted } \mathcal{D}\text{-mods} \\ \text{on } G/B \end{array} \right\}$$

D-modules:

$X/\mathbb{C}$ smooth alg var.

$$\mathrm{Vect}(X) = \mathrm{Der}(\mathcal{O}_X) = \{ A : \mathcal{O}_X \to \mathcal{O}_X \mid A(fg) = A(f)g + fA(g) \}$$

$\mathbb{C}$ - linear.

If $t_1, \dots t_n$ is a local coordinate system locally $\mathrm{Vect}(X) = \bigoplus_{i=1}^{n} \mathcal{O}_X \partial t_i$

$\mathcal{O}_X$ - Lie algebra

$\mathcal{D}_X$ = sheaf of differential operators = $\langle \mathrm{Vect}(X) \rangle$.

$\mathcal{D}_X$ is a sheaf of algebras on X. has a filtration by order of diff. operators.

$\mathrm{gr}\,\mathcal{D}_X \overset{*}{=} \pi_* \mathcal{O}_{T^*X}$, $\pi : T^*X \to X$.

Another characterisation of $\mathcal{D}_X$:

$$\mathcal{D}_X^{\leq r} = \{ A \mid \mathcal{O}_X \to \mathcal{O}_X \ , \ \text{s.t.:} \quad [\cdots[[A, x_0]\, x_1] \cdots x_n] = 0. \}$$

$\forall\ x_0, x_1, \cdots x_n \in \mathcal{O}_X$

$[A, x_0] = Ax_0 - x_0 A.$

e.g.: $n=0.$

$$\mathcal{D}_X^{\leq 0} = \{ A \mid [A, x] = 0 \quad \forall x \in \mathcal{O}_X \} = \mathcal{O}_X$$

$$\mathcal{D}_X^{\leq 1} = \mathcal{O}_X \cup \mathrm{Vect}(X) \qquad \text{(exercise).}$$

<u>Def 2</u>: $\mathcal{D}_X = \bigcup_{n=0}^{\infty} \mathcal{D}_X^{\leq n}.$ Question: Why are these two equivalent?

Let $\mathcal{L}$ be a line bundle on X. Using analogue of def 2 for $A: \mathcal{L} \to \mathcal{L}$.

We get a <u>twisted sheaf of diff. operators</u>

$$\mathcal{D}_X^{\mathcal{L}} \subseteq \mathrm{End}_{\mathbb{C}}(\mathcal{L}).$$

<u>Examples</u>: $\mathcal{L} = \mathcal{O}(n)$ on $\mathbb{P}^1$.

If $n=0$. z is the usual coordinate on $\mathbb{P}^1$. $w = 1/w$ on $\mathbb{P}^1 \setminus 0 = V$

$\mathbb{P}^1 \setminus \infty = U$

Normally. $\partial_\omega = -z^2 \partial_z$.

$\mathcal{L}(U) = \mathbb{C}[z]$ $\qquad\qquad$ $\mathcal{L}(V) = \mathbb{C}[w]$

Restrict to $U \cap V$ gluing datum: $\quad \mathbb{C}[z, z^{-1}] \xrightarrow[\simeq]{\varphi} \mathbb{C}[\omega, \omega^{-1}]$

$1 \longmapsto \omega^n$

$z \longmapsto z \cdot \varphi(1) = z \cdot \omega^n = \omega^{-1} \omega^n = \omega^{n-1}$.

$\partial_z (z^a) = a z^{a-1}$

$\varphi(z^a) = w^{n-a}$

$\varphi(a z^{a-1}) = a \omega^{n-a+1}$

what operator send ω^{n-a} to $a\omega^{n-a+1}$?

Answer: $n\omega - \omega^2 \partial\omega$.

$\partial_z = n\omega - w^2 \partial \omega$

More generally. G = semi simple simply connected group $/\mathbb{C}$

$B \supseteq T$ Borel, torus.

$X = G/B$. $\quad \mathrm{Pic}(G/B) = \mathrm{Hom}(T, \mathbb{G}_m) = X^*(T)$. $\Big\}$ Use simply connected condition.

Furthermore, $\mathrm{Pic}^G(G/B) = X^*(T)$.

For all $\lambda \in X^*(T)$. $\rightsquigarrow$ $\mathcal{D}^\lambda_{G/B}$

Rules: 1) often people use δ-shifts.

2) This construction generalizes to $\lambda \in \mathfrak{h}^* = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C} = H^2(X; \mathbb{C})$

Clear by computation when $G = SL_2$.

Some rmks: If G is <u>not</u> simply conn. $\mathrm{Pic}^G(G/B) =$ not described as above?

<u>BB Thm:</u>

center

$\lambda \in \mathfrak{h}^*$. $\rightsquigarrow$ $\theta_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ Central char. $:=$ char. of h.wt simple module with h.wt λ.

$Z(\mathfrak{g}) = Z(U(\mathfrak{g}))$

$U_\lambda := U(\mathfrak{g}) / \theta_\lambda$.

Suppose $\langle \lambda, \alpha_i^\vee \rangle \notin \{-1, -2, -\}$ $\forall$ simple α_i

Then: $\exists$ an equivalence of categories

$$\boxed{\{ D^\lambda_{G/B}\text{- modules} \} \underset{\Delta = D^\lambda_{G/B} \otimes_{U_\lambda} -}{\overset{\Gamma(G/B, -)}{\rightleftarrows}} \{ U_\lambda\text{- modules} \}}$$

localization functor

add: quasi-coherent.

<u>Examples:</u>

$G = SL_2$, $\lambda = 0$. $X = \mathbb{P}^1$

U_λ-modules :

$\Gamma(\mathbb{P}^1, \mathcal{O}) = 1$- dim.

$\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1}$ $\longrightarrow$ • trivial. 1-dimensional.

$\langle \delta_0 \rangle$ $\longrightarrow$ • Verma $M(-2)$ h. wt -2

Dirac δ- function at 0 in $\mathbb{P}^1$. • dual Verma $M^*(0)$ h. wt 0.

$z\,\delta_0 = 0$.

This D-mod is supported at 0.

A basis of its global sections is $\{\partial_z^n \delta_0\}_{n=0}^{\infty}$.

= weight vector in $M(-2)$.

Verma. $M(0)$ coresp. to an extension of $\mathcal{O}_{X_S}$ on X_S to $\mathbb{P}^1 = \{0\} \cup X_S$.

$U = \mathbb{P}^1 \setminus \infty$. $V = \mathbb{P}^1 \setminus 0$. allow a function to have a pole at 0.

Coordinate: z. w.

$M(U) = \mathbb{C}[z, z^{-1}]$ $M(V) = \mathbb{C}[w]$

$= \mathbb{C}[w, w^{-1}]$.

$\Gamma(\mathbb{P}^1, M) =$ meromorphic functions with poles at 0.
$= \mathbb{C}[w]$

This is the dual verma.

This M has a submodule gen. by w.

$$0 \to \mathcal{O}_X \hookrightarrow M \to (\delta_0) \to 0$$

dual Verma

Dual doesn't have such a nice description.

Dual verma:

$$0 \to \text{triv} \to M(0)^* \to M(-2) \to 0.$$

simple Verma

Example:

G arbitrary. $\quad \lambda \longleftrightarrow \mathcal{L}(\lambda)$ arbitrary.

$D^{\lambda}_{G/B} \curvearrowright \mathcal{L}$ D-module

$\Gamma(G/B, \mathcal{L}(\lambda)) = V_\lambda \leftarrow$ h. wt $\mathfrak{g}$-module h. wt $= \lambda$

Borel weil.

This example shows the conditions on λ are necessary

for an abelian equivalence.

Comment: There is a derived BB.

How do we relate $\mathfrak{g}$ to $D^{\Omega}_{G/B}$?

Theorem: $\boxed{\mathcal{U}_{\Omega} \cong \Gamma(D^{\Omega}_{G/B})}$

(Japanese).
HTT

There should be a natural map $\mathfrak{g} \to \Gamma(D^{\Omega}_{G/B})$.

- $\mathcal{L}$ is G-equivariant,

So if $X \in \mathfrak{g}$. can define $\partial_X : \mathcal{L} \to \mathcal{L}$ by:

$$(\partial_X \underset{\text{section}}{s})(y) = \frac{d}{dt}(e^{tX}s)(e^{-tX}y)\Big|_{t=0}$$

This defines $U(\mathfrak{g}) \to \Gamma(D^{\Omega}_{G/B})$

if $\mathfrak{g} = sl_2$.

$$\left.\begin{array}{l} e \longmapsto -\partial_w \\ f \longmapsto -\partial_z \end{array}\right\} \text{ indep. of } \Omega = \Omega$$

Use: $[e,f] = h$ + previous computation relating ∂_z & ∂_w.
about computing action of h.

At the level of assoc. graded $X = G/B$

$$\Gamma(X, \mathrm{gr}\, D_X^\lambda) \cong \Gamma(X, \pi_* \mathcal{O}_{T^*X})$$
$$= \Gamma(T^*X, \mathcal{O}_{T^*X})$$
$$= \Gamma(\mathcal{N}) \leftarrow \text{Nilcone.}$$
$$= S(\mathfrak{g}) \Big/ \langle S(\mathfrak{g})_+^G \rangle ?$$

has the same size as U_λ

$$\begin{array}{c} T^*X \\ \downarrow \\ \mathcal{N} \hookrightarrow \mathfrak{g}^* \end{array}$$

$$\begin{array}{c} T^*X \\ \downarrow \\ \mathcal{N} \end{array} \leftarrow \text{Nullcone.}$$

$S(\mathfrak{g}) = \mathrm{gr}\, U(\mathfrak{g})$.

How to check you have the right central char:

- $\Gamma(X, D^\lambda_{G/B})^G$: 1-dim.
- Look at action on $V_\lambda = \Gamma(X, \mathcal{L})$.

D-affinity:

Main ingredient of the proof

Need to be quasi-coherent.

Requires $H^i(X; M) = 0$, for all D_X^λ-module M.

Question. Is H^i defined in $\mathbb{C}_X$-mod, or $\mathcal{O}_X$-mod or $\mathrm{Qcoh}(X)$ or does it matter?

M quasi-Coh.

$M \cong \varinjlim N$, N coherent $\mathcal{O}_X$-module.

Suffices to show:

if $N \hookrightarrow M$ is a coherent submod.

then: $H^i(X, N) \to H^i(X, M)$ is 0 (for $i > 0$)

$$
\begin{tikzcd}
N \otimes^{(\mu)} V_\mu \arrow[r] & M \otimes V_\mu \\
N \otimes \mathcal{L}' \arrow[u] \arrow[r, "\text{twist}"'] & M \otimes \mathcal{L}' \arrow[u]
\end{tikzcd}
$$

V_μ - simple G-module

$\rightsquigarrow$ trivial v. bundle on X.

B-module filtration gives filtration on V_μ.

Some Translation functors

μ large enough

$H^i(-) = 0$

(Serre vanishing uses N coherent)

This arrow splits in $\mathcal{O}_X$-mod by $H^i(\cdot\cdot)$ central char argument.

$$
\begin{tikzcd}
0 \arrow[r] & H^i(\cdot\cdot) \\
H^i(N) \arrow[u] \arrow[r, "0"] & H^i(M) \arrow[u]
\end{tikzcd}
$$

Dual Vermas go to extensions of $\mathcal{O}_{X_w}$. $X_w \overset{i_w}{\hookrightarrow} X$ is a Schubert cell.

$$\int_{i_w} \mathcal{O}_{X_w} = i_{w*} \left(D^{\lambda}_{X \leftarrow X_w} \otimes_{D^{\lambda}_{X_w}} \mathcal{O}_{X_w} \right)$$

- Dual Vermas are (inductively) characterise them by
- check character
- show that the D-module has no submodules supported on smaller strata.

No homomorphism

dual Verma wt λ $\longrightarrow$ dual Verma wt μ. with $\lambda < \mu$.

Using the dot action.

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It doesn't matter in which category you define $H^i(X, -)$

$\mathcal{F}$ an injective $\mathcal{O}_X$-mod (Pf: Consider $j_{U!}\mathcal{O}_U \hookrightarrow j_{V!}\mathcal{O}_V$

apply $\mathrm{Hom}(-, \mathcal{F})$.

$\Downarrow$

$\mathcal{F}$ is flasque (if $U \subseteq V$ open, then $\mathcal{F}(V) \twoheadrightarrow \mathcal{F}(U)$ surj.)

$\Downarrow$

$\mathcal{F}$ is acyclic for Γ.

D-modules:

A D-module is always (by def) quasi-coherent as an $\mathcal{O}_X$-mod.

G simply conn. $\lambda \in \mathrm{Hom}(T, \mathbb{G}_m)$.

simple alg group $/\mathbb{C}$,

$\rightsquigarrow$ (twisted) sheaf of diff. operators D^λ on G/B.

B-B: If $\langle \lambda, \alpha_i^\vee \rangle \geq 0 \ \forall i$

$$\{D^\lambda\text{-mod}\} \cong \{U_\lambda\text{-mod}\}$$

Major point of the proof:

1) • Γ is exact
2) • $\Gamma(M) = 0 \Rightarrow M = 0$
3) • $\Gamma(D^\lambda) = U_\lambda$

About 1): V a reps of G. W a B-submodule of V

Define: $\mathcal{E}_W = \{(gB, gw) \subseteq G/B \times V \mid g \in G, w \in W\}$

$\mathcal{E}_W \downarrow G/B$ is a vector bundle

In particular, $V = V_\mu$ of highest wt μ.

gives $\mathcal{O}(\mu) \hookrightarrow V_\mu$ ← trivial bundle.

Global section functor has a left adjoint

$\Delta = D^\lambda \otimes_{U_\lambda} -$ — localization functor ← turns out to be the inverse equiv. to Γ.

Let $M \in U_\Omega$-mod. It has a presentation:

$$\begin{tikzcd} U_\Omega^{\oplus J} \arrow[r] \arrow[d, "\wr"] & U_\Omega^{\oplus I} \arrow[r] \arrow[d, "\wr"] & M \arrow[r] \arrow[d] & 0 \\ \Gamma\circ\Delta(U_\Omega^{\oplus J}) \arrow[r] & \Gamma\circ\Delta(U_\Omega^{\oplus I}) \arrow[r] & \Gamma\circ\Delta(M) \arrow[r] & 0 \end{tikzcd}$$

$\therefore$ $M \to \Gamma\circ\Delta(M)$ is an isom (use Γ is exact).

$M \in D^\Omega$-mod.

kernel, cokernel

$$0 \to \cancel{K} \to \Delta\Gamma M \to M \to \cancel{Q} \to 0$$

$$0 \to \cancel{\Gamma K} \to \Gamma\Delta\Gamma M \xrightarrow{\cong} \Gamma M \to \cancel{\Gamma Q} \to 0$$

$\Rightarrow \Delta\Gamma M \cong M$.