Proof of K-L conjecture

Reference Springer Bourbaki, 82

I. K-L conjecture & Hecke algebra

$\mathfrak{g}$ = semisimple complex Lie algebra

$\cup\!|$

$\mathfrak{b}$ = Borel

$\cup\!|$

$\mathfrak{h}$ = Cartan $\quad \rho \in \mathfrak{h}^*$ $\quad \rho: \mathfrak{h} \to \mathbb{C}$

takes value 1 at each simple coroot

$$M_w = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\underbrace{-w(\rho)-\rho}_{\text{highest wt}}} \quad \text{Verma module}$$

$\downarrow$

L_w = the unique irreducible quotient

Easy: $\{[M_w]\}_{w \in W}$ & $\{[L_w]\}_{w \in W}$ basis of $K^0(\mathcal{O}_0)$

W = Weyl group

conj [Kazhdan-Lusztig 79]

$$[L_w] = \sum_{y \le w} \varepsilon_y \varepsilon_w P_{y,w}(1)\, [M_y]$$

$$[M_y] = \sum_{y \le w} P_{w_0 w, w_0 y}(1)\, [L_y]$$

Kazhdan-Lusztig polynomials

$\varepsilon_w = (-1)^{\ell(w)}$, $\quad w_0$ = longest element

$$\mathcal{H} = \mathbb{Z}[q^{\pm\frac{1}{2}}]\langle T_w \rangle_{w \in W} \Big/ \begin{array}{l} \cdot\ T_w T_{w'} = T_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w') \\ \cdot\ (T_s + 1)(T_s - q) = 0 \text{ if } s \in S \end{array}$$

S = the set of reflections

Observations: $\cdot$ $\{T_w\}$ form a basis,

• relation 2 $\Rightarrow$ $T_s^{-1} = q^{-1} T_s + (q^{-1} - 1)$

<u>Defn</u> (bar involution)

$$\beta : \mathbb{Z}[q^{\pm\frac{1}{2}}] \longrightarrow \mathbb{Z}[q^{\pm\frac{1}{2}}]$$
$$q^{\frac{1}{2}} \longmapsto q^{-\frac{1}{2}}$$
$$\beta : \mathcal{H} \longrightarrow \mathcal{H}$$
$$T_w \longmapsto T_{w^{-1}}^{-1}$$

NOTE! This is an algebra homomorphism

<u>Thm</u> [Kazhdan-Lusztig]

$\forall w \in W$, $\exists!$ $C_w \in \mathcal{H}$ s.t. ① $\beta C_w = C_w$

② $C_w = q^{-\frac{1}{2}\ell(w)} \sum P_{y,w}(q) T_y$

where $P_{y,w} \in \mathbb{Z}[q^{\frac{1}{2}}]$

$\deg \leq \ell(w) - \ell(y) - 1$

& $P_{w,w} = 1$

$$\boxed{q^{-\frac{1}{2}\ell(w)} = q^{-\frac{1}{2}(\ell(w)-\ell(y))} \cdot q^{-\frac{1}{2}\ell(y)}}$$

Moreover • $P_{y,w} = 0$ if $y \not\leq w$

• $P_{y,w} \in \mathbb{Z}[q]$

<u>Examples</u> • $C_1 = T_1 = 1$

• $C_s = q^{-\frac{1}{2}}(T_s + 1)$

$\beta(T_s + 1) = T_s^{-1} + 1 = q^{-1} T_s + q^{-1}$

<u>Rmk</u> • Thm $\Leftrightarrow$ Thm$'$:

$\exists!$ B_w s.t. • $\beta(B_w) = B_w$

• $B_w = \sum \varepsilon_y \varepsilon_w q^{-\frac{1}{2}\ell(w)} q^{-\ell(y)} P_{y,w}(q^{-1}) T_y$

$P_{y,w}$ as before

• True for any Coxeter group]

• (later) $P_{y,w} \in \mathbb{N}[q^{\frac{1}{2}}]$

• Also, Thm follows from the geometric method

• Two parts of K-L conjecture are equivalent

Can be shown algebraically: $\sum_{x \le z \le y} \varepsilon_x \varepsilon_y P_{xz} P_{w_0 y, w_0 z} = \delta_{x,y}$

Kari may have comments

I. Sheaves on flag varieties

recap: upto some q-powers & sign's
L_w in terms of M_w (conj)
$= C_w \quad \cdots \quad T_w$

$$\begin{array}{ccccccc}
\mathcal{O}_S & \xleftrightarrow{\text{BB-localization}} & \mathcal{D}_{\mathcal{B}}\text{-mod} & \xleftrightarrow{\text{R-H}} & D^b(\mathcal{B}) & \xleftrightarrow{\text{sheaf-function}} & \mathcal{H} \\
M_w & & \text{Local cohomology } H^{d-\ell(w)}_{[X_w]}(X, \mathcal{O}_X) & & \underline{\mathbb{C}}_{X_w}[\ell(w)] & & q^{-\frac{\ell(w)}{2}} T_w \\
L_w & & \text{irreducible} & & IC(\overline{X_w}) & & C_w
\end{array}$$

<u>Recall</u> X smooth, $\mathcal{S} = \{S_i\}$ $\quad \coprod S_i = X$ stratification (whitney conditions)

$$IC(S_i) \in D^b_{\mathcal{S}}(X)$$

is characterized by that $IC(S_i)_x = \mathbb{C}$ if $x \in S_i$

$x \in S_j \qquad H^k(IC(S_i)_x) = 0$ unless $\quad S_j \subseteq \overline{S_i}$
$-\dim S_i \le k \le -\dim S_j - 1$

$$\boxed{\begin{array}{c} \dim \operatorname{supp} \underline{H}^{-k}(IC(S_i)) \le k-1 \\ \text{unless } k=n \ \& \ \operatorname{supp} \underline{H}^{-n}(\ ) = X \\ \underline{H}^{-k} = 0 \text{ for } k \le 0 \end{array}}$$

Reminder on Schubert varieties

$B \curvearrowright \mathcal{B} = G/B \qquad \mathcal{B} = \coprod_{w\in W} \underbrace{B\cdot wB/B}_{=: X_w}$

<u>Facts</u>: • $\overline{X_w} = \coprod_{v \le w} X_v$ schubert variety

• $B\backslash G/B \xleftrightarrow{1:1} G\backslash \mathcal{B}\times\mathcal{B}$

$$\mathcal{B}\times\mathcal{B} = \coprod_{w\in W} \underset{\mathcal{O}(w)}{\underset{\|}{G\cdot\left(B/B, wB/B\right)}}$$

$$\begin{tikzcd}
\overline{\mathcal{O}(w)} \cong G\times_B \overline{X_w} \arrow[d] & \supseteq & \mathcal{O}(w) = G\times_B X_w \arrow[d, "\text{locally trivial fibration}"] \\
\overline{X_w} & \supseteq & X_w
\end{tikzcd}$$

<u>Example</u>: • $X_1 = pt$, $\quad \mathcal{O}(1) = G/B \overset{\Delta}{\hookrightarrow} \mathcal{B}\times\mathcal{B}$

$s \in S$ • $X_s = \begin{array}{c}\mathbb{A}^1 \\ \cup \\ p\end{array}$, $\quad \overline{X_s} = B/B \sqcup B\cdot s\cdot B/B = P_s/B \cong \mathbb{P}^1$

$$\begin{tikzcd}
\overline{\mathcal{O}(s)} \cong G\times_B \overline{X_s} \arrow[d] \arrow[r, "\mathbb{P}^1\text{-fibration}"] & G/B \\
\overline{X_s} & \mathcal{B}\times_{\mathcal{P}_s}\mathcal{B} \arrow[u]
\end{tikzcd}$$

$G\times_B \overline{X_s} \cong \mathcal{B}\times_{\mathcal{P}_s}\mathcal{B}$, $\quad \mathcal{P}_s = G/P_s$

<u>In particular</u> $\forall y \in W$

$$IC(\overline{X_w})_y \cong IC(\overline{\mathcal{O}(w)})_y[\dim \mathcal{B}]$$

$y \in X_y \qquad y \in \mathcal{O}(y)$

· Bott-Samelson resolution

$$\begin{array}{c} Y_w = P_{s_1} \times_B P_{s_2} \times_B \cdots \times_B P_{s_q} / B \\ \downarrow \text{resolution of singularities} \\ \mathcal{B} \supseteq \overline{X_w} \end{array} \qquad w = s_1 s_2 \cdots s_q \text{ reduced word}$$

(isom if $l(w) \le 2$)

· similarly $\mathcal{Y}_w := G \times_B Y_w$ $\qquad \mathcal{B}_s = G/P_s$

$$\mathcal{Y}_w \cong \{ (a_0, \dots, a_q) \mid (a_{i-1}, a_i) \in \overline{O(s_i)} \} \cong \mathcal{B} \times_{\mathcal{B}_{s_1}} \mathcal{B} \times_{\mathcal{B}_{s_2}} \cdots \times_{\mathcal{B}_{s_q}} \mathcal{B}$$

$$\underbrace{\mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}}_{q+1} \supseteq \{ (a_0, \dots, a_q) \} \xrightarrow{\pi} \mathcal{B} \times \mathcal{B} \supseteq \overline{O(w)} \ni (a_0, a_q)$$

Construction:

$$\begin{array}{ccc} A^\bullet \in D^b_B(\mathcal{B}) & \rightsquigarrow & h(\underline{A}^\bullet) \in \mathcal{H} \\ \downarrow & & \shortparallel \\ \underline{A}^\bullet \in D^b_G(\mathcal{B} \times \mathcal{B}) & & \sum_{w \in W} \left(\sum_i \dim \underline{H}^i(\underline{A}^\bullet)_w \, q^{\frac{i}{2}} \right) T_w \end{array}$$

Example:

- $IC(X_1) = \underline{\mathbb{C}}_{pt}, \quad h(IC(X_1)) = \underline{1}$
- $IC(\overline{X_s}) = \underline{\mathbb{C}}_{\overline{X_s}}[1]$
- $IC(\overline{O(s)}) = \underline{\mathbb{C}}_{\overline{O(s)}}[\dim \mathcal{B} + 1]$

· $h(IC(\overline{X_s})) = q^{-\frac{1}{2}} T_s + q^{-\frac{1}{2}} \cdot \underline{1} \quad = C_s$

Rmk: Does not directly factor through the Grothendieck group

could $q^{-\frac{1}{2}} = -1$.

will NOT do this until the last step

Key Lemma

- $\underbrace{\underline{A}^\bullet, \underline{B}^\bullet}_{\mathcal{B}} \in D^b_G(\mathcal{B}\times\mathcal{B})$

$$\underline{A}^\bullet * \underline{B}^\bullet := q_! \, i^* \left(p_{12}^* \underline{A}^\bullet \otimes p_{34}^* \underline{B}^\bullet \right)$$

$$\begin{tikzcd}
& \mathcal{B}\times\mathcal{B}\times\mathcal{B}\times\mathcal{B} \arrow[dl, "p_{12}"'] \arrow[dr, "p_{34}"] & & \mathcal{B}\times\mathcal{B}\times\mathcal{B} \arrow[ll, hook', "id_1\times\Delta_{23}\times id_4"'] \arrow[d, "q = pr_{13}"] \\
\mathcal{B}\times\mathcal{B} & & \mathcal{B}\times\mathcal{B} & \mathcal{B}\times\mathcal{B}
\end{tikzcd}$$

Associative in the usual sense

$s \in S$ $\quad \underline{A}_s := \underline{\mathbb{C}}_{\overline{O(s)}} \in D^b_G(\mathcal{B}\times\mathcal{B})$

Lemma If $\underline{A}^\bullet \in D^b_G(\mathcal{B}\times\mathcal{B})$ is such that $\underline{H}^i(\underline{A}^\bullet) = 0$ for i even/odd

then so is $\underline{A}_s * \underline{A}^\bullet$

$$\& \quad h(\underline{A}_s * \underline{A}^\bullet) = (T_s + 1)\cdot h(\underline{A}^\bullet)$$

consequently $h(\underline{A}_{s_1} * \underline{A}_{s_2} * \cdots * \underline{A}_{s_q}) = h(\underline{A}_{s_1}) \cdots h(\underline{A}_{s_q})$

$$= (T_{s_1}+1) \cdots (T_{s_q}+1)$$

Note: $Y_w = \mathcal{B} \times_{\mathcal{B}_{s_1}} \mathcal{B} \times_{\mathcal{B}_{s_2}} \cdots \times_{\mathcal{B}_{s_q}} \mathcal{B} \subseteq \overbrace{\mathcal{B}\times\cdots\times\mathcal{B}}^{q+1}$

$$\begin{tikzcd}
Y_w \arrow[d, "\pi"] \\
\overline{O(w)}
\end{tikzcd}$$

$$R\pi_* \underline{\mathbb{C}}_{Y_w} = \underline{A}_{s_1} * \cdots * \underline{A}_{s_q}$$

COR 1 $\quad h(R\pi_* \underline{\mathbb{C}}_{Y_w}) = (T_{s_1}+1)\cdots(T_{s_q}+1)$

Observe $\quad \mathbb{D}$ = Verdier dual

$$h(\mathbb{D} R\pi_* \underline{\mathbb{C}}_{Y_w}) = h(R\pi_* \underline{\mathbb{C}}_{Y_w}[2\ell(w)]) = q^{-\ell(w)} (T_{s_1}+1)\cdots(T_{s_q}+1)$$

$$= \beta\left((T_{s_1}+1)\cdots(T_{s_q}+1)\right)$$

COR 2 If $A^{\cdot} \in D^{b}_{G_F}(\mathcal{B}\times\mathcal{B})$ is of the form $\bigoplus_{\substack{i \in \mathbb{Z} \\ w \in W}} V_{i,w} \otimes R\pi_{*} \underline{\mathbb{C}}_{y_w}$

Then $h(\mathbb{D}(A^{\cdot})) = \beta\, h(A^{\cdot})$

Rmk So far the coefficients of sheaves are arbitrary

- Let $\mathcal{D} \in D^{b}_{B}(\mathcal{B})$ be semi-simple objects

i.e. $\bigoplus_{w,i} IC(\overline{X_w}) \otimes V_{w,i}$

$$\begin{array}{ccc} Obj(\mathcal{D})^{(\text{mond})} & & \\ \downarrow & \searrow & \\ G(\mathcal{D}) & \longrightarrow & \mathcal{H} \end{array}$$

$G(\mathcal{D})$ = Split Grothendieck group.

- $Im[Obj(\mathcal{D})] \in \mathbb{N}[q^{\pm\frac{1}{2}}]\{T_w\}$

- $G(\mathcal{D})_{q=1} \cong K^{0}(D^{b}_{B}(\mathcal{B}))$

- Using <u>Decomposition thm</u>, $obj(\mathcal{D})$ is closed under $*$

$(G(\mathcal{D}), *)$ is an $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra

<u>Trivial</u>:
- $G(\mathcal{D})^{1} :=$ subalgebra gen. by $IC(\overline{O(s)})$

$G(\mathcal{D})^{1} \longrightarrow \mathcal{H}$

is an ring homomorphism

$$\left[\text{recall } h(IC(\overline{X_s})) = q^{-\frac{1}{2}}(T_s + 1)\right]$$

- $G(\mathcal{D})^{2} :=$ sub alg. gen by $R\pi_{*}\underline{\mathbb{C}}_{y_w}$

$G(\mathcal{D})^{2} \longrightarrow \mathcal{H}_{q}$ commutes

$\circlearrowleft_{\mathbb{D}} \qquad \circlearrowleft_{\beta}$

A closer look

$\mathcal{Y}_w$ is smooth of dim $\ell(w)$

$$IC(\mathcal{Y}_w) = \underline{\mathbb{C}}_{\mathcal{Y}_w}[\ell(w)]$$

$$\xrightarrow{\text{Decomposition thm}} R\pi_* \underline{\mathbb{C}}_{\mathcal{Y}_w}[\ell(w)] \cong \bigoplus_{\substack{x \le w \\ i \in \mathbb{Z}}} IC(\overline{O(x)}) \otimes V_{x,i}$$

In particular $V_{x,i} = 0$ if $x \not\le w$

& $V_{w,i} = 0$ if $i \ne 0$

$$\| $$

$$IC(\overline{O(w)}) \oplus \left(\bigoplus_{x < w} IC(\overline{O(x)}) \otimes V_{x,i} \right)$$

Observation: Induction on the Bruhat order

$$G(D)^1 = G(D)^2 = G(D)$$

MAIN Thm $h(IC(\overline{O(w)})) = C_w$

Pf By defn of IC-sheaf

$$H^i(IC(\overline{O(w)})_y) \ne 0 \text{ only if } i \in [-\ell(w), -\ell(y)-1]$$

& by observation $\beta h(IC(\overline{O(w)})) = h\mathbb{D}(IC(\overline{O(w)}))$

$$= h(IC(\overline{O(w)})) \quad \square$$

Rmk $R\pi_* \underline{\mathbb{C}}_{\mathcal{Y}_w}[\ell(w)] \cong \bigoplus_{\substack{x \le w \\ i \in \mathbb{Z}}} IC(\overline{O(x)}) \otimes V_{x,i}$

$$\xrightarrow{\mathbb{D} \circ R\pi_* = R\pi_* \circ \mathbb{D}} V_{x,i} \cong V_{x,-i}$$

Apply h: LHS $\| $ RHS $\|$

$$q^{-\frac{\ell(w)}{2}} (T_{s_1}+1) \cdots (T_{s_q}+1) = h(IC(\overline{O(w)})) + \sum_{x < w} Q_x \cdot \underbrace{h(IC(\overline{O(x)}))}_{\substack{\text{induction} \\ C_x}}$$

$$Q_x \in \mathbb{N}[q^{\pm\frac{1}{2}}] \ \& \ Q_x = \beta Q_x$$

This also shows $h\, IC(\overline{O(w)})$ is invariant under β

COR ① The existence of C_w w/ prescribed properties

$$C_w = \sum_{y\le w} q^{-\frac{\ell(w)}{2}} P_{y,w}\, T_y \qquad \text{(KL Thm)}$$

② $[L_w] = \sum (-1)^{\ell(w)-\ell(y)} P_{y,w}(1)\,[M_y]$

③ $P_{y,w} \in \mathbb{N}[q]$

Infact $P_{y,w}(q^{\frac12}=-1)$ but only depends on q

Proof of Key Lemma

$$H^i(\underline{A}^\bullet)=0 \text{ if } i \text{ odd}$$

$\Rightarrow$ same for $IC(\overline{O(s)}) * \underline{A}^\bullet$

& $h\left(IC(\overline{O(s)}) * \underline{A}^\bullet\right) = (1+T_s)\,h(\underline{A}^\bullet)$

$$\boxed{(1+T_s)T_w = \begin{cases} T_{sw}+T_w, & \ell(sw)>\ell(w) \\ q(T_w+T_{sw}) & \ell(sw)<\ell(w)\end{cases}}$$

Suffices to show

$$\dim \underline{H}^i\left(\underline{\mathbb{C}}_{\overline{O(s)}} * \underline{A}^\bullet\right)_w = \begin{cases} \dim \underline{H}^i(\underline{A}^\bullet)_{sw} + \dim \underline{H}^{i-2}(\underline{A}^\bullet)_w & \text{if } sw<w \\ \dim \underline{H}^i(\underline{A}^\bullet)_w + \dim \underline{H}^{i-2}(\underline{A}^\bullet)_{sw} & \text{if } sw>w \end{cases}$$

$\underline{\mathbb{C}}_{\overline{O(s)}} \boxtimes \underline{A}^\bullet$

$$\begin{array}{ccccc} \mathcal{B}\times\mathcal{B}\times\mathcal{B}\times\mathcal{B} & \xleftarrow{i} & \mathcal{B}\times\mathcal{B}\times\mathcal{B} & \xrightarrow{p_{13}} & \mathcal{B}\times\mathcal{B} \\ \cup\!\! | & & \cup & & \cup\!\! | \\ \overline{O(s)}\times\mathcal{B}\times\mathcal{B} & & \overline{O(s)}\times\mathcal{B} & & O(w) \ni (1,w) \\ \| & & & & \\ O(w)\amalg O(sw) & & & & \end{array}$$

$D := \overline{O(s)}\times\mathcal{B} \cap p_{13}^{-1}(1,w) \supset \{(1, gB/B, wB/B) \mid g\in P_s\}$

$\cong \mathbb{P}^1$

$$H^i\left((\underline{\mathbb{C}}_{\overline{O(s)}} * \underline{A})_w\right) \cong H^i\left(i^*\left(\underline{\mathbb{C}}_{\overline{O(s)}} \boxtimes \underline{A}^\bullet\right)\Big|_D\right)$$

Assume $sw < w$. Look for $\overline{O(s)} \times B \cap p_{23}^{-1} O(w) \overset{\sim}{=} G\cdot(sB/B, wB/B)$ $\cap D$

& $\overline{O(s)} \times B \cap p_{23}^{-1} O(sw)$

$gB \neq sB$

$\mathbb{A}^1$

$(gB/B, wB/B) \in G\cdot(B/B, wB/B)$

$g \in P_s \Rightarrow gB = sB$

only 1 pt

$$pt \xhookrightarrow{i} \mathbb{P}^1 \xhookleftarrow{j} \mathbb{A}^1$$

$$\mathcal{C}$$

$$i^* \mathcal{C}|_{pt} \cong \underline{A}_{sw}$$

$$j^* \mathcal{C}|_{pt} \cong \underline{A}_w$$

$$\cdots H_c^{i-1}(i_* i^* C) \to H_c^i(j_! j^* C) \to H_c^i(C) \to H_c^i(i_* i^* C) \to H_c^{i+1}(j_! j^* C) \to \cdots$$

$H_c^{i-1}(i_* i^* C) = 0$, $H_c^i(j_! j^* C) \cong H^{i-2}(\underline{A}_w)$, $H_c^i(i_* i^* C) \cong H^i(\underline{A}_{sw})$, $H_c^{i+1}(j_! j^* C) = 0$

Assume $sw > w$

$$\left(\overline{O(s)} \times B \cap p_{23}^{-1} O(w)\right) \cap D = pt$$

$$\left(\quad \cdots \quad sw\right) \cap D = \mathbb{A}^1$$

similar to the above $\square$