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Localization of Representations of Reductive Lie Algebras

In this report I shall sketch the algebro-geometrical viewpoint on the representations of real reductive groups. One will see that representation theory deals with things quite familiar to algebraic geometers: namely with the geometry of Schubert-like varietes. As for applications, I shall confine myself to the following ones: a classification of irreducible representations [1], [13], the Kazhdan–Lusztig character formulas [7], [10] and the structure of Jantzen's filtration (the degeneration of series of representations). The last two subjects are based on arithmetical considerations (the theory of mixed perversed sheaves [3], [4]).

All the schemes considered will be over a fixed algebraically closed ground field k of char 0.

A. Affine spaces and localization

Let X be a scheme.

DEFINITION. An *$\mathcal{O}_X$-ring* is a sheaf $\mathcal{R}$ of rings on X, together with a ring morphism $\mathcal{O}_X \to \mathcal{R}$ such that $\mathcal{R}$ is quasicoherent as a left $\mathcal{O}_X$-module. For an $\mathcal{O}_X$-ring $\mathcal{R}$, an $\mathcal{R}$-module is a sheaf of left $\mathcal{R}$-modules, quasicoherent as a sheaf of $\mathcal{O}_X$-modules. ∎

Denote by $\mathcal{R}$-mod the category of $\mathcal{R}$-modules. Put $R := \Gamma(X, \mathcal{R})$. There are natural adjoint functors $\mathcal{R}\text{-mod} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} R\text{-mod}$: $\Gamma(\mathcal{M}) := \Gamma(X, \mathcal{M})$, $\Delta(N) = \mathcal{R} \underset{R}{\otimes} N$, and also corresponding derived functors $R\Gamma$ and $L\Delta$.

DEFINITION. We shall say that X is *$\mathcal{R}$-affine* if Γ and Δ are (mutually inverse) equivalences of categories; and that X is *$\mathcal{R}$-affine in the sense of derived categories of amplitude* $\leqslant n$ if $R\Gamma$ is equivalence of derived categories of amplitude $\leqslant n$. ∎

If for a ring R there exist some $(X, \mathscr{R})$ such that $R = \Gamma(X, \mathscr{R})$ and X is $\mathscr{R}$-affine, then one may study R-modules by local methods as sheaves over X; in this situation we call the sheaf $\Delta(N)$ the *localization* of an R-module N.

Here is a criterion for $\mathscr{R}$-affinity. Any $\mathscr{R}$-module $\mathscr{M}$ is generated by global sections and $H^i(X, \mathscr{M}) = 0$ for $i > 0$.

The $\mathcal{O}_X$-rings that we shall consider are the rings of differential operators $\mathscr{D}_X$ on some smooth X, or slightly more general rings introduced in the next section. Here is a somewhat striking example (see Section C): any flag space X (e.g. $X = \boldsymbol{P}^N$) is $\mathscr{D}_X$-affine.

B. Twisted rings of differential operators

Let X be a smooth variety.

Definition. An $\mathcal{O}_X$-ring is called a *ring of twisted differential operators* (tdo for short) if it is locally isomorphic (on X), as an $\mathcal{O}_X$-ring, to $\mathscr{D}_X$. If A is any commutative k-algebra, then an A-tdo is a sheaf of A-algebras, and also an $\mathcal{O}_X$-ring, locally isomorphic to $\mathscr{D}_X \underset{k}{\otimes} A$. ■

Remark. A-tdo is just a family of tdo's, parametrized by $\operatorname{Spec} A$.

As the automorphisms of $\mathcal{O}_X$-ring $\mathscr{D}_X$ are exactly closed 1-forms on X (a 1-form ω corresponds to an automorphism $\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \omega\left(\frac{\partial}{\partial x}\right)$), one may identify tdo's with $\Omega_X^{1,\mathrm{cl}}$-torsors (and A-tdo's with $A \otimes \Omega_X^{1\mathrm{cl}}$-torsors). In particular tdo's form a "linear k-space". If χ is a torsor, we denote the corresponding tdo by $\mathscr{D}_X^{\chi}$. The set of isomorphic classes of tdo is $H^1(X, \Omega_X^{1,\mathrm{cl}})$, for a proper X it is the k-subspace of $H^2_{DR}(X)$ generated by algebraic cycles.

Examples. (a) If $\mathscr{L}$ is any invertible $\mathcal{O}_X$-sheaf, then the sheaf $\mathscr{D}_{X,\mathscr{L}}$ of differential operators on $\mathscr{L}$ is a tdo. One has $\mathscr{D}_{X,\mathscr{L}} = \mathscr{D}_X^{\mathrm{dlog}\mathscr{L}}$.

(b) If $\mathscr{D}$ is tdo, then $\mathscr{D}^0$ – a ring with inverted multiplication – is also a tdo. If χ is an $\Omega_X^{1,\mathrm{cl}}$-torsor, put $\chi^0 := d\log \Omega - \chi$, where Ω is the sheaf of volume forms $\det \Omega_X^1$. One has $(\mathscr{D}_X^{\chi})^0 = \mathscr{D}_X^{\chi^0}$, as Ω has the canonical structure of a *right* $\mathscr{D}_X$-module.

C. Main construction

Let G be a connected reductive group over k, $\mathscr{G} := \operatorname{Lie} G$ its Lie algebra, $U = U(\mathscr{G})$ the universal enveloping algebra, and Z the centre of U. Let X be the flag manifold of G (the space of its Borel subgroups), and T_X the sheaf of vector fields on X. For $x \in X$ let $B_x \supset N_x$ be the corresponding Borel subgroup and its radical, $H := B_x/N_x$ the Cartan group (it does not depend on x), and $b_x \supset n_x$ and h the corresponding Lie algebras. There is a Harish-Chandra map $Z \subset S(h)$ which identifies Z with the space of W-invariant polynomials on h^* (where the Weyl group W acts fixing ϱ — the half-sum of positive roots), let Θ: Spec $S(h) \to$ Spec Z be the corresponding map. Let $\tilde{U} := U \underset{Z}{\otimes} S(h)$ be the extended universal enveloping algebra. It is clear that the centre of $\tilde{U}$ is $S(h)$.

Now we pass to the construction. The group G acts on X and one has the corresponding Lie algebra map $a\colon \mathscr{G} \to T_X$. Define an $\mathcal{O}_X$-ring structure on $U_X := \mathcal{O}_X \underset{k}{\otimes} U$ by demanding that $[A, f] = a(A)f$ for $A \in \mathscr{G} \subset U, f \in \mathcal{O}_X$ and that the multiplication on $U \subset U_X$ should coincide with the usual one. Consider the induced Lie algebra structure on $\mathscr{G}_X := \mathcal{O}_X \otimes \mathscr{G} \subset U_X$, and put $\mathscr{B}_X = \ker(a\colon \mathscr{G}_X \to T_X) = \{\xi \in \mathscr{G}_X\colon \forall x \in X\, \xi(x) \in b_x\}$, $\mathscr{N}_X = [\mathscr{B}_X, \mathscr{B}_X] = \{\xi \in \mathscr{G}_X\colon \forall x \in X\ \xi(x) \in n_x\}$. These are ideals in $\mathscr{G}_X$. Put $\tilde{\mathscr{D}} := U_X/U_X \cdot \mathscr{N}_X$. One has obvious maps $U \to \Gamma(X, \tilde{\mathscr{D}})$, and $S(h) \to \Gamma(X, \tilde{\mathscr{D}})$ (as $\mathcal{O}_X \otimes h = \mathscr{B}_X/\mathscr{N}_X \subset U_X$). It is easy to see that $S(h)$ is mapped onto the centre of $\tilde{\mathscr{D}}$, and both these maps coincide on Z. So they define the map $\tilde{U} \to \Gamma(X, \tilde{\mathscr{D}})$.

LEMMA. (a) *This map is an isomorphism*: $\tilde{U} = \Gamma(X, \tilde{\mathscr{D}})$.
(b) $\tilde{\mathscr{D}}$ *is an* $S(h)$*-tdo on* X. ∎

To apply this to the study of $\mathscr{G}$-modules we need to verify the affinity of the picture. For simplicity we confine ourselves to Z-finite $\mathscr{G}$-modules, i.e., $\mathscr{G}$-modules annihilated by some ideal of finite codimension in Z. Let $\chi \in h^*$ be a character, and $m_\chi \subset S(h)$ the corresponding maximal ideal. Put $\widehat{S(h)}_\chi := \varprojlim S(h)/m_\chi^n$, $\mathscr{D}_\chi := \tilde{D}/m_\chi \tilde{\mathscr{D}}$, $\hat{\mathscr{D}}_\chi := \tilde{\mathscr{D}} \underset{S(h)}{\otimes} \widehat{S(h)}_\chi$. It is clear that $\mathscr{D}_\chi$ is a tdo (and $\hat{\mathscr{D}}_\chi$ is an $\widehat{S(h)}_\chi$-tdo), and one has $\Gamma(X, \mathscr{D}_\chi) =: U_\chi = \tilde{U}/m_\chi \tilde{U}, \Gamma(X, \hat{\mathscr{D}}_\chi) =: \hat{U}_\chi = \hat{U} \otimes \widehat{S(f)}_\chi$.

Remark. If χ is integral, i.e., originates in the character of H, then $\mathscr{D}_\chi$ is a sheaf of differential operators on a corresponding invertible sheaf on X. In particular $\mathscr{D}_0 = \mathscr{D}_X$.

Now suppose χ to be regular. Then Θ: Spec $S(h) \to$ Spec Z is étale at χ and so $\widehat{S(h)}_\chi = \hat{Z}_{\Theta(\chi)}$, $\hat{U}_\chi = \hat{U}_{\Theta(\chi)} := U \underset{Z}{\otimes} \hat{Z}_{\Theta(\chi)}$, $U_\chi = U/m_{\Theta(\chi)} U$. This means that $\hat{U}_\chi(\hat{U}_{\Theta(\chi)})$-modules are just $\mathscr{G}$-modules with a (generalized) central character $\Theta(\chi)$. Denote by Γ_χ, Δ_χ the corresponding functors $\overset{(\wedge)}{D}_\chi\text{-mod} \rightleftarrows \overset{(\wedge)}{U}_{\Theta(\chi)}-\text{mod}$.

Recall that $\chi \in h^*$ is dominant if for any simple positive coroot l one has $\chi(l) \neq 0, -1, \ldots$

THEOREM [1], [2]. *Let χ be a regular weight. If χ is dominant then X is $\mathscr{D}_\chi$- and $\hat{\mathscr{D}}_\chi$-affine. If $\chi = w\chi_0$ where χ_0 is dominant, then X is $\mathscr{D}_\chi$- and $\hat{\mathscr{D}}_\chi$-affine in the sense of the derived categories, of amplitude $\leqslant$ the length of w.* ∎

So $\mathscr{G}$-modules with a regular central character θ are just $\mathscr{D}_\chi$-modules for dominant χ such that $\Theta(\chi) = \theta$.

Remark. Let M be a U_θ-module and $x \in X$. One knows that the spectrum of the natural action of h on $H.(n_x, M)$ is contained in $\Theta^{-1}(\theta)$: put $H. = \bigoplus_{\chi \in \Theta^{-1}(\theta)} H._\chi$. The definition implies that $H.(n_x, M)_\chi = \mathrm{Tor.}^{\mathcal{O}_x}(k_x, L\Delta_\chi M)$ — the fibre at x of $L\Delta_\chi M$ in the sense of $\mathcal{O}_X$-modules. So the theorem is a generalization (to arbitrary $\mathscr{G}$-modules) of the highest-weight and Borel–Weil–Bott theory of finite-dimensional representations.

D. Functorial properties of $\mathscr{D}$-modules

Here we recall some basic functors on $\mathscr{D}$-modules (following Bernstein, Kashiwara, ...) and show what these functors mean for representations.

D1. Translation. Let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-sheaf and χ any torsor; then $\mathscr{D}^\chi_X$-mod is canonically equivalent to $\mathscr{D}^\chi_{X,\mathscr{L}}$ $(= \mathscr{D}_X^{\chi + d\log\mathscr{L}})$-mod: one transforms the $\mathscr{D}^\chi$-module $\mathscr{M}$ to $\mathscr{L} \underset{\mathcal{O}_X}{\otimes} \mathscr{M}$ with a canonical $D^\chi_{X,\mathscr{L}}$-action. So for any regular dominant $\chi_1, \chi_2 \in h^*$ such that $\chi_1 - \chi_2$ is integral one has a canonical equivalence between the categories $\overset{(\wedge)}{U}_{\theta(\chi_i)}$-mod; this is the Bernstein–Gelfand translation principle.

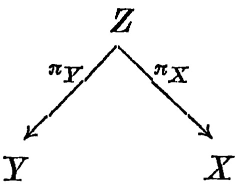
D2. Action of correspondences. Let $f: Y \to X$ be a morphism of smooth varietes, and $\mathscr{D}_{(X)} = \mathscr{D}^\chi_X$ be a tdo on X; then one has a tdo $\mathscr{D}_{(Y)} := D^{f^{-1}(\chi)}$ on Y. Define exact functors $f^!$: $\mathscr{D}(\mathscr{D}_{(X)}\text{-mod}) \to \mathscr{D}(\mathscr{D}_{(Y)}\text{-mod})$ and f_*: $\mathscr{D}(\mathscr{D}_{(Y)}\text{-mod}) \to \mathscr{D}(\mathscr{D}_{(X)}\text{-mod})$ between derived categories as follows. First, for any $\mathscr{D}_{(X)}$-module $\mathscr{M}$ there is a natural $\mathscr{D}_{(Y)}$ action on $f^+(\mathscr{M}) := \mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \mathscr{M}$ — the inverse image of $\mathscr{M}$ in the sense of $\mathcal{O}$-modules. Let Lf^+: $\mathscr{D}(\mathscr{D}_{(X)}\text{-mod})$

$\to \mathscr{D}(\mathscr{D}_{(Y)}$-mod) be a derived functor of f^+; put $f^! := Lf^+[\dim Y - \dim X]$. To define f_* one uses $(\mathscr{D}_{(Y)} - f^{\cdot}(\mathscr{D}_{(X)}))$-bimodule structure on $f^+(\mathscr{D}_{(X)})$: put $f_*(\mathscr{N}) = Rf.(\mathscr{N} \overset{L}{\underset{\mathscr{D}_{(Y)}}{\otimes}} f^+(\mathscr{D}_{(X)}))$.

Examples. (a) If f is a closed imbedding, then f_* and $f^!$ induce equivalence between $\mathscr{D}_{(Y)}$-mod and the category of $\mathscr{D}_{(X)}$-modules supported on Y.

(b) If f is smooth, then $f_*(\mathscr{M})$ is the relative de Rham complex with coefficients in $\mathscr{M}$, shifted by the relative dimension of f.

Remark. These functors may be unified by considering correspondences. Namely, consider the diagram



where Z is also smooth, and two torsors, χ_X on X and χ_Y on Y, with a fixed isomorphism $\pi_Y^{-1}(\chi_Y^0) = \pi_X^{-1}(\chi_X^0)$. Let Z_*: $\mathscr{D}(\mathscr{D}_Y^{\chi_Y}$-mod$) \to \mathscr{D}(\mathscr{D}_X^{\chi_X}$-mod) be $\pi_{X*}\pi_Y^!$.

Now let us return to representations. Let X be as in Section C and $w \in W$. We have the Bruhat–Hecke correspondence $N_w = \{(x, x') \in X \times X$ such that $(b_x, b_{x'})$ are in relative position $w\}$.

THEOREM [2]. *Let χ be dominant regular. Then* $(N_w)_* L\Delta_\chi = L\Delta_{w(\chi)}$. ∎

So the action of the Bruhat–Hecke correspondences on $\mathscr{D}$-modules (the intertwining functors of [2]) corresponds to the action of W on $\tilde{U}$. These correspondences play a very important role in the Kazhdan–Lusztig theory (of Section H).

D3. Duality. Let $\mathscr{D}$ be a tdo. One says that $\mathscr{D}$-module is *coherent* if it is locally finitely generated; let $\mathscr{D}$-modc be the category of coherent modules. Consider $\mathscr{D}$ as $(\mathscr{D}, \mathscr{D})$-bimodule; it defines a natural functor $*$: $\mathscr{D}(\mathscr{D}$-modc$)^0 \to \mathscr{D}(\mathscr{D}^0$-modc) by the formula $*\mathscr{M} := R\,\mathrm{Hom}(\mathscr{M}, \mathscr{D}[\dim X])$; one has $** = \mathrm{id}$. Coherent modules correspond to finitely generated representations, and $*$ corresponds to the functor $*M := R\mathrm{Hom}(M, U_\theta[\dim X])$.

Example. One says that a $\mathscr{D}$-module is smooth if it is coherent as a sheaf of $\mathcal{O}_X$-modules, or, equivalently, if after a (local) isomorphism $\mathscr{D} \simeq \mathscr{D}_X$ it becomes a sheaf of sections of a bundle with integrable connection. For smooth $\mathscr{M}$ we have $*\mathscr{M} = \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{M}, \Omega_X)$ with an obvious $\mathscr{D}^0$-module structure.

E. Holonomic modules and the Harish-Chandra modules

The holonomic $\mathscr{D}$-modules are those that are smooth along a certain stratification. Let me introduce one important construction before giving the exact definition. Consider the affine locally closed imbedding $i: Y \hookrightarrow X$, where Y is smooth, and a smooth $\mathscr{D}_{(Y)}$-module $\mathscr{M}$. Then $i_*\mathscr{M}$ is a coherent $\mathscr{D}_{(X)}$-module. Put $i_! \mathscr{M} := *i_* *\mathscr{M}$; we have $i^! i_! \mathscr{M} = i^! i_* \mathscr{M} = \mathscr{M}$ and there is a unique morphism $\varphi: i_!\mathscr{M} \to i_*\mathscr{M}$ such that $i^!(\varphi) = \mathrm{id}_{\mathscr{M}}$. Denote $\mathrm{Im}\,\varphi$ by $i_{!*}\mathscr{M}$. If $\mathscr{M}$ is irreducible, then $i_{!*}\mathscr{M}$ is the unique irreducible submodule of $i_*\mathscr{M}$ (and the unique irreducible quotient of $i_!\mathscr{M}$). In this situation the modules $i_*\mathscr{M}$, $i_!\mathscr{M}$ are called *standard modules* and $i_{!*}\mathscr{M}$ is called the *irreducible module* corresponding to $(Y, \mathscr{M})$.

By definition a $\mathscr{D}$-module $\mathscr{M}$ is holonomic if it has a finite length and all of his Jordan–Hölder components are of the type constructed above. One says that holonomic $\mathscr{M}$ (on compact X in the twisted case) has regular singularities (RS's for short) if all its components originate in bundles with regular singularities at infinity. The basic property of holonomic modules is that the corresponding derived category of complexes with holonomic cohomology is stable under the functors of the type $f^!, f_*$; if $\mathscr{M}$ is holonomic, then $*\mathscr{M}$ is also holonomic. The same applies for holonomic RS's.

Now return to $\mathscr{G}$-modules. The representation theorists claim that the representations that happen in nature are $(\mathscr{G}, K)$-modules for a certain (algebraic) subgroup $K \subset G$ (cf. [13]).

Roughly speaking, a $(\mathscr{G}, K)$-module N is $\mathscr{G}$-module s.t. the action of Lie K may be integrated to an algebraic representation of K. Indeed, one simply fixes the algebraic action of K on N with obvious compatibility conditions. It is easy to see that $(\mathscr{G}, K)$-modules correspond to $(\mathscr{D}, K)$-modules on X, i.e., to $\mathscr{D}$-modules with such an action of K that Lie K acts via the imbedding Lie $K \hookrightarrow \mathscr{D}$.

To get an interesting theory one needs sufficiently large subgroups K. Say that K is admissible if Lie K is transverse to some Borel subalgebra or, equivalently, if K acts on X with finitely many orbits. Fix an admissible K. It is easy to see that any coherent $(\mathscr{D}, K)$-module is smooth along the orbits of K and is holonomic RS. The irreducible $(\mathscr{D}, K)$-modules are in a 1–1 correspondence by means of the $i_{!*}$ construction, with irreducible smooth $(\mathscr{D}_{(Y)}, K)$-modules on different K-orbits Y, and these are defined by representations of the stabilizers of points.

Arrange this to get a classification of irreducible $(\mathscr{D}, K)$ (and so of $(\mathscr{G}, K)$)-modules. For any K-orbit Y put $H_Y := K \cap B_y/_{K \cap N_y} \subset H$ where

$y \in Y$ (note that H_Y does not depend on $y \in Y$); $h_Y := \mathrm{Lie} H_Y$, and thus H_Y is a product of the torus H_Y^0 and the finite abelian group H_Y/H_Y^0.

THEOREM [1], [13]. *For $\chi \in h^*$, the irreducible $(\mathscr{D}_\chi, K)$-modules are in a 1–1 correspondence with the set of pairs (Y, τ_Y), where Y is a K-orbit on X and τ_Y is the irreducible (h, H_Y)-module on which h acts by χ. If χ is regular dominant, this is also the classification of irreducible $(\mathscr{G}, K)$-modules with central character $\Theta(\chi)$.* ■

Remarks. (a) We see that the standard and irreducible modules that correspond to an orbit Y form $\dim(h/_{h_Y})$-parameter families (integrality condition on $\chi/_{h_Y}$). All the standard modules are irreducible for generic values of the parameter. If they are irreducible for any value of the parameters (or for some integral one) then Y is a closed orbit — this is the case of (generalized) discrete series. Some information on the non-closed orbits case will be presented in Section I. (In fact, to work clearly with the standard modules one has to suppose that for any orbit its imbedding into X is affine; this is the case in any example of the next remark.)

(b) The subgroups K usually considered are the fixed points of involutions of G (this correspond to the Harish-Chandra modules or representations of real reductive groups) or either N_x or B_x for certain $x \in X$ (the representations with highest weights); the second case may be reduced to the first (the representations of complex reductive groups). Such subgroups K are admissible. The standard modules for $K = N$ are just the Verma modules and the dual ones; in the Harish-Chandra situation the standard modules (and thus the classification) coincide with those of Langlands. This was proved by Vogan in [12]: he compared the above construction with that of Zuckerman (the cohomological parabolic induction). It would be interesting to use some analysis to compare it directly with that of Langlands. I understand that J. Bernstein has been working on this subject.

(c) The standard modules are considered as the simplest from the representation-theoretic point of view; in particular in Harish-Chandra's situation their characters are known. As both the standard and the irreducible modules form bases in the Grothendieck group of $(\mathscr{G}, K)$-modules, it is of importance to find the ones in terms of the others. And this is what the Kazhdan–Lusztig algorithm does.

F. Perverse sheaves

Now suppose that our ground field is $\boldsymbol{C}$. Let us describe the topological interpretation of holonomic RS $\mathscr{D}$-modules. In the case of smooth modules what follows is just the old result of Deligne, which claims that the local systems are the same as bundles with integrable connections having RS at infinity.

Let X be a smooth variety and $\mathscr{M}$ a $\mathscr{D}_X$-module. Denote by $\Omega(\mathscr{M})$ the analytic de Rham complex of $\mathscr{M}$: $\Omega(\mathscr{M}) := (\mathscr{M} \to \Omega^1 \underset{\mathcal{O}_X}{\otimes} \mathscr{M} \to \ldots)$, $\mathscr{M}$ being placed in degree $-\dim X$. This defines the functor Ω from the derived category of $\mathscr{D}_X$-modules to that of sheaves on X_{an}. Kashiwara proved that $\Omega(\mathscr{M})$ has a constructible cohomology for holonomic $\mathscr{M}$. And, due to Kashiwara, Kawai and Mebkhout, we have the following comparison theorem.

THEOREM. *The functor Ω induces an equivalence between the derived category of the complexes of $\mathscr{D}_X$-modules, having a holonomic RS cohomology, and that of the complexes of C-sheaves on X, having a constructible cohomology. This equivalence transforms $*$ to the Verdier duality, and $f_*, f^!$ to the functors of the same notation from the constructible sheaf theory.* ■

Example. Return to the situation from the beginning of Section E; suppose that $\mathscr{M}$ has RS. Then $\Omega(\mathscr{M})$ is the local system, corresponding to $\mathscr{M}$, placed in degree $-\dim Y$. We have $\Omega(i_!\mathscr{M}) = i_!\Omega(\mathscr{M})$, the same applies to i_*, and $\Omega(i_{!*}\mathscr{M})$ is the Deligne–Goresky–MacPherson complex of $\Omega(\mathscr{M})$ on $\overline{Y}$, prolonged by 0 on $X \setminus \overline{Y}$ ([1], [5]). ■

But what about $\mathscr{D}_X$-modules themselves? If $\mathscr{M}$ is a holonomic RS, then according to Kashiwara we have the following conditions on $\Omega(\mathscr{M})$: $\dim \operatorname{supp} H^i(\Omega(\mathscr{M})) \leqslant -i$ for any i, and the same applies to $*\Omega(\mathscr{M})$. Such constructible complexes are called *perverse sheaves* [3], [4], and the theorem implies that Ω induces the equivalence between the category of $\mathscr{D}_X$-modules holonomic RS and that of perverse C-sheaves.

Let me show how to deal with the tdo case. To be brief, consider the flag space only. There is a canonical H-torsor π: $\tilde{X} = G/N \to X = G/B$ over X ("the base affine space"). The ring $\pi_*(\mathcal{O}_{\tilde{X}})$ is graded by the weight lattice, and we have the corresponding gradation on $\pi_*\mathscr{D}_{\tilde{X}}$. The zero component of $\pi_*\mathscr{D}_{\tilde{X}}$ is just $\tilde{\mathscr{D}}$, so $\tilde{\mathscr{D}}$-modules are the same as graded $\pi_*\mathscr{D}_{\tilde{X}}$-modules — and so are $\mathscr{D}_{\tilde{X}}$-modules. This leads to the following perverse description of $\mathscr{D}_\chi$- and $\hat{\mathscr{D}}_\chi$-modules. A perverse sheaf $\mathscr{F}$ on $\tilde{X}$ is said to be *monodromic* if it is smooth along the fibres of π. For such $\mathscr{F}$ one has the monodromy representation (along the fibres) of the co-weight lattice $(= \pi_1(H))$ in $\operatorname{Aut}\mathscr{F}$ [11]. Then the category of (holonomic RS) $\mathscr{D}_\chi$-modules is equivalent to that of monodromic perverse C-sheaves on $\tilde{X}$ of monodromy $\exp\chi$ and $\hat{\mathscr{D}}_\chi$-modules are monodromic sheaves such that every eigenvalue of monodromy is $\exp\chi$.

Remark. The advantage of replacing holonomic modules by perverse sheaves is the possibility to use any coefficient ring (e.g., one has the $\mathcal{Q}_l$-Harish-Chandra modules). Since $\mathcal{Q}_l$-perverse sheaves has an étale meaning, this opens the door to arithmetics (certainly, there should be an arithmetical crystalline theory of $\mathcal{D}$-modules, but at a moment one has to use constructible sheaves).

G. Motivic language: mixed perverse sheaves

The yoga of motives claims that there should exist a fine category of motives over X such that any "natural" perverse sheaf on X or $\mathcal{D}_X$-module is a realization of a certain motive. Since the motivic theory has not appeared yet, one is forced to use its l-adic realization — the theory of mixed perverse sheaves [3], [4], based on the Weil conjectures proved by Deligne [6].

Here are a few properties of mixed sheaves. There is an abelian category $\mathcal{M}_{\text{mixed}}(X)$ of mixed perverse sheaves on X, together with canonical functor from $\mathcal{M}_{\text{mixed}}(X)$ to the category $\mathcal{M}(X)$ of $\mathcal{Q}_l$-perverse sheaves. Any mixed perverse sheaf M has a canonical (finite) decreasing weight filtration $W.(M)$; any morphism in $\mathcal{M}_{\text{mixed}}$ is strongly compatible with the weight filtration. The object $\mathrm{Gr}_{W.}(M)$ is semi-simple, at least in $\mathcal{M}(X)$. There is Verdier's duality functor $*$ on $\mathcal{M}_{\text{mixed}}$, compatible with the one on $\mathcal{M}(X)$; one has $*W_i(M) = W_{-i}(*M)$. There is also the corresponding derived category of mixed sheaves, together with all the standard functors, compatible with the one on usual sheaves.

Thus, in this way we get the category of mixed Harish-Chandra modules (this category will be non-empty for rational χ, but, if that is so, then any irreducible module has a mixed structure).

Problem. Construct this category by representation theoretical means (any natural representation should get a mixed structure: and so the weight filtration with the properties above).

I shall mention two applications of this mixed category. The first is the Kazhdan–Lusztig algorithm (the starting point of all the things above) and the second one is the structure of Jantzen's filtration on standard modules.

H. The Kazhdan–Lusztig algorithm

What follows is only a very rough exposition of the basic ideas; the algorithm itself may be found in [7], [10] (the Verna modules case) and in [12], [13] (the general Harish-Chandra case, due to Vogan).

Suppose we are given a variety X stratified by strata X_j and also a number of irreducible local systems (= smooth perverse sheaves) $\{V_j\}$ on any X_j. Suppose that every $i_j\colon X_j \hookrightarrow X$ is affine, and that irreducible components of any $i_!(V_j)$ are isomorphic to some $i_{j'!*}(V_{j'})$. Let $\mathscr{U}$ be a subgroup of the Grothendieck group of perverse sheaves on X generated by $\{i_{j!*}(V_j)\}$. It has two natural bases: the irreducible one $\{i_{!*}(V)\}$ and the standard one $\{i_!(V)\}$. The problem is to compute $\{i_{!*}\}$ via $\{i_!\}$ (cf. Section E, Remark c).

To do this, suppose that everything has arisen from a mixed situation (and so we have $\{V\}_{\text{mixed}}, \mathscr{U}_{\text{mixed}}, \ldots$), and that the set $\{V\}_{\text{mixed}}$ is $*$-closed. Then $*$ acts on $\mathscr{U}_{\text{mixed}}$. We have one extra structure on $\mathscr{U}_{\text{mixed}}$ — namely the weight filtration $W_\cdot$. The space W_j may be defined in terms of standard bases, as the one generated by all $i_!(V)$ such that V is of weight $\leqslant j$. Then, if the weight of V is j, then $i_{!*}(V)$ is the unique element of $\mathscr{U}_{\text{mixed}}$, such that $i_{!*}(V) \in W_j \cap *W_{-j}$ and $i_{!*}(V) - i_!(V) \in W_{j-1}$. So to compute $\{i_{!*}\}$ in terms of $\{i_!\}$ it suffices to know the matrix of $*$ in $i_!$-bases.

To find $*$ in the representation-theoretic situation, one uses the action of the Hecke algebra (already appearing in the theorem of Section D2 — but now we need its mixed variant). The Hecke algebra $\mathscr{H}$ is the $\mathscr{U}_{\text{mixed}}$-group related to the stratification of $X \times X$ by G-orbits (and the constant sheaves). The multiplication on $\mathscr{H}$ is the multiplication of correspondences. This algebra — the mixed variant of the group algebra of the Weyl group — may be given explicitly by generators and relations, and the $*$ operator on $\mathscr{H}$ is given by compatibility with multiplication and by an explicit formula on generators [7], [8]. Let us return to Harish-Chandra's modules. The Hecke algebra acts on the corresponding $\mathscr{U}_{\text{mixed}}$ group, and the $*$ operator on $\mathscr{U}_{\text{mixed}}$ is more or less determined by compatibility with this action and by the claim that $i_!(V) = i_{!*}(V)$ for closed orbits [9], [12].

I. The Jantzen's filtration

This filtration shows how the standard representations become reducible under the specialization of parameters.

Return to the situation of Section E. Fix an orbit Y, the character $\chi \in h^*$, integral with respect to Y, and the (h, H_Y)-module τ of the theorem quoted there. We have the corresponding standard modules $i_!(\tau)$, $i_*(\tau)$ and the canonical morphism $i_!(\tau) \to i_*(\tau)$, whose image is the irreducible $i_{!*}(\tau)$. Now we may vary τ in the family having $(f/h_Y)^* + \tau \subset f^*$ as parameters (see Remark (a) of Section E) to obtain the corresponding $\hat{\mathscr{D}}_\chi$-modules $i_!(\hat{\tau})$ and $i_*(\hat{\tau})$ together with the morphism $i_!(\hat{\tau}) \to i_*(\hat{\tau})$.

From now on we suppose that Lie $K = \mathscr{G}^\sigma$ for certain involution σ of $\mathscr{G}$ (see Remark (b) of Section E; the case $K = N_x$ is even simpler). Let C be the intersection of $(f/h_Y)^*$ with the positive cone of rational characters. If Y is not closed, then C is non-zero. Choose φ in the open cone C^0 and consider the one-parameter subfamily τ_φ of $\hat{\tau}$ that depends on $\chi + t\varphi$. One knows that $i_!(\tau_\varphi) \to i_*(\tau_\varphi)$ is an injection whose cokernel $i_*/i_!(\tau_\varphi)$ is of finite length.

Define the filtration $I^{(\varphi)}_{\bullet}$ on $i_*(\tau)/i_{!*}(\tau) = i_*/i_!(\tau_\varphi)/ti*/i!(\tau)(\tau_\varphi)$ by the formula $J^{(\varphi)}_n := [\mathrm{Ker}(t^n \in \mathrm{End}\ (i_*(\dot{\tau}_\varphi)/_{i!x}(\tau_\varphi))]\mathrm{mod}\, t$; this defines the filtration $J^{(\varphi)}_{\bullet}$ on $i_*(\tau)$ such that $i^{(\varphi)}_0 = i_{!*}(\tau)$.

This is Jantzen's filtration.

Remark. One may show, that the standard $\mathscr{G}$-module $i_!(\tau)$ is the K-finite dual to a certain $i_*(\tau')$; so Jantzen's filtration may be defined in terms of a "contravariant form".

THEOREM. *The filtration $I^{(\varphi)}$ coincides, up to a shift, with the weight filtration on the mixed perverse sheaf $i_*(\tau)$.* ∎

COROLLARY. *The filtration $I^{(\varphi)}_{\bullet}$ does not depend on the choice of $\varphi \in C^0$. The module $\mathrm{gr}_{J_\cdot^{(\varphi)}}(i_*(\tau))$ is a direct sum of irreducible ones. The multiplicities in terms of Jantzen's filtration are given by the Kazhdan–Lusztig–Vogan algorithm.* ∎

To prove the theorem one has to identify the module $i_*/i_{!*}(\tau_\varphi)$ with certain sheaf of vanishing cycles, and then to use Gabber's purity theorem for vanishing cycles.

The theorem above, due to J. Bernstein and the present author, was conjectured (in the Verma modules case) by Brylinski; the corresponding numerical statement is the generalized Kazhdan–Lusztig conjecture of Gabber–Joseph and S. Gelfand–MacPherson.

References

[1] Beilinson A. and Bernstein J., Localisation de $\mathscr{G}$-modules, *C. R. Acad. Sc. Paris* **292**, pp. 15–18.

[2] Beilinson A. and Bernstein J., A Generalisation of a Theorem of Casselman, *Report on Utah Conference on Representation Theory, April, 1982.*

[3] Beilinson A., Bernstein J., and Deligne P., Faisceaux perverse, *Astérisque* **100** (1982).

[4] Brylinski J. L., (Co)homologie d'intersection et faisceaux pervers, *Sem. Bourbaki exposé* **585**, fevrier 1982.

[5] Brylinski J. L. and Kashiwara M., Kazhdan–Lusztig Conjecture and Holonomic Systems, *Inv. Math.* **64** (1981), pp. 387–410.

[6] Deligne P., La conjecture de Weil II, *Publ. Math. IHES* **52** (1980), pp. 137–252.

[7] Kazhdan D. and Lusztig G., Representation of Coxeter Groups and Hecke Algebras, *Inv. Math.* **53** (1979), pp. 165–184.
[8] Kazhdan D. and Lusztig G., Schubert Varietes and Poincaré Duality, *Proc. Symp. Pure Math.* **36**, pp. 185–203.
[9] Lusztig G. and Vogan D., Singularities of Closures of K-Orbits on Flag Manifolds, *Inv. Math.* **71** (1983), pp. 365–380.
[10] Springer T., Quelques applications de la cohomologie d'intersection, *Sem. Bourbaki exposé* **589**, fevrier 1982.
[11] Verdier I. L., Specialisation de faisceaux et monodromie moderée, *Astérisque* **101–102** (1983), pp. 332–364.
[12] Vogan D., Irreducible Characters of Semisimple Lie Groups III. Proof of Kazhdan–Lusztig Conjecture in the Integral Case, *Inv. Math.* **71** (1983), pp. 380–418.
[13] Vogan D., *Representation of Real Algebraic Groups*, Birkhäuser, 1981.

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