ARTIN ALGEBRAIZATION FOR PAIRS WITH APPLICATIONS TO THE LOCAL STRUCTURE OF STACKS AND FERRAND PUSHOUTS

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ABSTRACT. We give a variant of Artin algebraization along closed subschemes and closed substacks. Our main application is the existence of étale, smooth, or syntomic neighborhoods of closed subschemes and closed substacks. In particular, we prove local structure theorems for stacks and their derived counterparts and the existence of henselizations along linearly fundamental closed substacks. These results establish the existence of Ferrand pushouts, which answers positively a question of Temkin–Tyomkin.

1. INTRODUCTION

The main technical result of this paper is a generalization of Artin's algebraization theorem [Art69, Thm. 1.6]: from algebraizations of complete local rings to algebraizations of rings complete along an ideal. It is proven using Artin approximation over henselian pairs following the approach of [CJ02] and [AHR20, App. A].

Theorem 1.1 (Artin algebraization for pairs)**.** *Let S be an excellent affine scheme and let $\mathscr{X}$ be a category fibered in groupoids, locally of finite presentation over S. Let Z be an affine scheme over S, complete along a closed subscheme Z_0. Assume that $Z_0 \to S$ is of finite type. Let $\eta \colon Z \to \mathscr{X}$ be a morphism, formally versal at Z_0. Then there exist*

(1) *an affine scheme W of finite type over S,*
(2) *a closed subscheme $W_0 \hookrightarrow W$,*
(3) *a morphism $\xi \colon W \to \mathscr{X}$ over S and*
(4) *a morphism $\varphi \colon (Z, Z_0) \to (W, W_0)$ over S*

such that the induced morphism $\widehat{\varphi} \colon Z \to \widehat{W}$ is an isomorphism and the isomorphism $\varphi_n \colon Z_n \to W_n$ on infinitesimal neighborhoods is compatible with η and ξ for every n.

We prove a more general version when Z is a stack in Theorem 2.3. This generalizes [AHR20, App. A] and is used to establish a local structure theorem for stacks (Theorem 1.3). We will return to this shortly.

Application: Étale neighborhoods of affine subschemes. As an application of Theorem 1.1 we have the existence of affine étale neighborhoods.

Theorem 1.2 (Affine étale neighborhoods). *Let $\mathscr{X}$ be a quasi-separated algebraic stack with affine stabilizers, and consider a diagram*

$$
\begin{array}{ccc}
W_0 & \lhook\!- - \to & W \\
\downarrow {\scriptstyle f_0} & & \vdots {\scriptstyle f} \\
\mathscr{X}_0 & \lhook\!\!\longrightarrow & \mathscr{X},
\end{array}
$$

where $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ is a closed immersion and $f_0 \colon W_0 \to \mathscr{X}_0$ is an étale (resp. smooth) morphism with W_0 affine. Then there exist an affine scheme W and an étale (resp. smooth) morphism $f \colon W \to \mathscr{X}$ such that $f|_{\mathscr{X}_0} = f_0$.

If $\mathscr{X}$ is an affine scheme, then Theorem 1.2 is [SP, 04D1] (f_0 étale) and [Elk73, Thm. 6] (f_0 smooth). For non-affine schemes and algebraic spaces, these results are new and answer positively a question of Temkin and Tyomkin [TT16, Qstn. 5.3].

Application: Local structure of stacks. We now generalize Theorem 1.2 from extending affine étale neighborhoods to extending linearly fundamental étale neighborhoods. By definition, an algebraic stack $\mathscr{X}$ is *fundamental* if there is an affine morphism $\mathscr{X} \to B\mathrm{GL}_{n,\mathbb{Z}}$ for some n, and *linearly fundamental* if it is fundamental and cohomologically affine; see [AHR19, §2.2] for further discussion.

In order to formulate mixed-characteristic versions of the local structure results, we recall from [AHR19, §15] the following conditions on an algebraic stack $\mathscr{X}$.

- (FC) There is only a finite number of different characteristics in $\mathscr{X}$.
- (PC) Every closed point of $\mathscr{X}$ has positive characteristic.
- (N) Every closed point of $\mathscr{X}$ has nice stabilizer [HR15, Defn. 1.1] (i.e., is an extension of a finite linearly reductive group scheme by an algebraic group of multiplicative type).

If $\mathscr{X}$ is linearly fundamental, then (PC) $\implies$ (N) as linearly reductive group schemes in positive characteristic are nice [Nag62], [HR15, Thm. 1.2]. The condition that we often impose will be of the following form for some morphism of stacks $\mathscr{W}_0 \to \mathscr{X}$: assume either that $\mathscr{W}_0$ satisfies (N), or $\mathscr{X}$ satisfies (FC).

We also remind the reader of another type of algebraic stack from [AHR19, §2.2]: an algebraic stack $\mathscr{X}$ is *nicely fundamental* if it admits an affine morphism to $B_S Q$, where $Q \to S$ is a nice and embeddable group scheme over S. It follows that nicely fundamental stacks are linearly fundamental.

Theorem 1.3 (Local structure of stacks). *Let S be an excellent algebraic space and let $\mathscr{X}$ be an algebraic stack, quasi-separated and locally of finite presentation over S with affine stabilizer groups. Consider a diagram*

$$
\begin{array}{ccc}
\mathscr{W}_0 & \lhook\!- - \to & \mathscr{W} \\
\downarrow {\scriptstyle f_0} & & \vdots {\scriptstyle f} \\
\mathscr{X}_0 & \lhook\!\!\longrightarrow & \mathscr{X},
\end{array}
$$

where $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ is a closed immersion and $f_0 \colon \mathscr{W}_0 \to \mathscr{X}_0$ is a morphism of algebraic stacks with $\mathscr{W}_0$ linearly fundamental.

(1) *If f_0 is smooth (resp. étale), then there exists a smooth (resp. étale) morphism $f \colon \mathscr{W} \to \mathscr{X}$ such that $\mathscr{W}$ is fundamental and $f|_{\mathscr{X}_0} \simeq f_0$.*

(2) *Assume that $\mathscr{W}_0$ satisfies (PC) or (N) or $\mathscr{X}_0$ satisfies (FC). If f_0 is syntomic and $\mathscr{X}_0$ has the resolution property, then there exists a syntomic morphism $f \colon \mathscr{W} \to \mathscr{X}$ such that $\mathscr{W}$ is fundamental and $f|_{\mathscr{X}_0} \cong f_0$.*

Syntomic means flat and locally of finite presentation, with fibers that are local complete intersections. An important example in our context is that any morphism $BG \to \mathscr{G}_x$ is smooth in characteristic zero but merely syntomic in positive characteristic.

For further refinements on $\mathscr{W}$, see Theorems 1.5 and 1.6 below and [AHR19, §16–17]. For a non-noetherian version, see Theorem 5.1. We also have the following result.

Theorem 1.4 (Local structure of stacks at non-closed points)**.** *Let $\mathscr{X}$ be a quasi-separated algebraic stack with affine stabilizer groups. Let $x \in |\mathscr{X}|$ be a point with residual gerbe $\mathscr{G}_x$ and let $f_0 \colon \mathscr{W}_0 \to \mathscr{G}_x$ be a syntomic (resp. smooth, resp. étale) morphism with $\mathscr{W}_0$ linearly fundamental. Then there exists a syntomic (resp. smooth, resp. étale) morphism $f \colon \mathscr{W} \to \mathscr{X}$ such that $\mathscr{W}$ is fundamental and $f|_{\mathscr{G}_x} \cong f_0$.*

We give a more general version for pro-affine-immersions in Theorem 5.8. Note that the inclusions $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ of a closed substack in Theorem 1.3 and $\mathscr{G}_x \hookrightarrow \mathscr{X}$ of a residual gerbe in Theorem 1.4 are both pro-affine-immersions. We also have refinements on the local charts (cf. [AHR19, Prop. 12.5 and Cor. 17.4]).

Theorem 1.5 (Refinement 1)**.** *Let $\mathscr{W}$ be a fundamental stack. Let $\mathscr{W}_0 \hookrightarrow \mathscr{W}$ be a pro-affine-immersion. Assume that $\mathscr{W}_0$ is linearly fundamental and satisfies (PC), (N), or (FC). If $g \colon \mathscr{W} \to \mathscr{X}$ is a morphism to an algebraic stack with affine (resp. separated) diagonal, such that $g|_{\mathscr{W}_0}$ is representable, then there exists an étale neighborhood $\mathscr{W}' \to \mathscr{W}$ of $\mathscr{W}_0$ such that $\mathscr{W}'$ is fundamental and $g|_{\mathscr{W}'}$ is affine (resp. representable).*

Theorem 1.6 (Refinement 2)**.** *Let $\mathscr{W}$ be a fundamental stack and $\mathscr{W}_0 \hookrightarrow \mathscr{W}$ be a pro-affine-immersion. Assume that $\mathscr{W}_0$ is linearly fundamental and that either $\mathscr{W}_0$ satisfies (PC), (N), or $\mathscr{W}$ satisfies (FC). Then there exists an étale neighborhood $\mathscr{W}' \to \mathscr{W}$ of $\mathscr{W}_0$ such that*

(1) *$\mathscr{W}'$ is linearly fundamental.*
(2) *If $\mathscr{W}_0 = [\operatorname{Spec} A_0/G_0]$, where G_0 is a linearly reductive (resp. nice) and embeddable group scheme over the good moduli space W_0, then we can arrange so that $\mathscr{W}' = [\operatorname{Spec} A/G]$, where G is a linearly reductive (resp. nice) and embeddable group scheme over the good moduli space W', such that $G|_{W_0} \cong G_0$.*

Application: Henselizations. The *henselization* of an algebraic stack $\mathscr{X}$ along a morphism $\nu \colon \mathscr{W} \to \mathscr{X}$ is an initial object in the 2-category of 2-commutative diagrams

$$
\begin{array}{ccc}
\mathscr{W} & \longrightarrow & \mathscr{X}' \\
& \nu \searrow & \downarrow f \\
& & \mathscr{X},
\end{array}
$$

where $f \colon \mathscr{X}' \to \mathscr{X}$ is pro-étale. Recall that $f \colon \mathscr{X}' \to \mathscr{X}$ is called *pro-étale* if it is an inverse limit of quasi-separated étale neighborhoods $\mathscr{X}_\lambda \to \mathscr{X}$ such that the transition maps $\mathscr{X}_\lambda \to \mathscr{X}_\mu$ are affine for all sufficiently large $\lambda \geq \mu$. Note that we do not require that f is representable or separated.

Theorem 1.7 (Existence of Henselizations)**.** *Let $\mathscr{X}$ be a quasi-separated algebraic stack with affine stabilizers. Let $\nu \colon \mathscr{X}_0 \hookrightarrow \mathscr{X}$ either be the inclusion of a closed substack satisfying (PC), (N), or (FC); or the inclusion of a residual gerbe. If $\mathscr{X}_0$ is linearly fundamental, then the henselization $\mathscr{X}_\nu^h$ of $\mathscr{X}$ along ν exists. Moreover, $\mathscr{X}_\nu^h$ is linearly fundamental and $(\mathscr{X}_\nu^h, \mathscr{X}_0)$ is a henselian pair.*

When $\mathscr{X}$ is an affine scheme, then Theorem 1.7 is [Ray70, Ch. XI, Thm. 2]. The result is new for non-affine schemes and algebraic spaces. It is also closely related to, but does not settle, conjectures of Greco and Strano on henselian schemes [GS81, Conj. A, B and C].

Note that there are no analogous results for open neighborhoods: there are schemes with affine closed subschemes that do not admit affine neighborhoods. Indeed, there is a separated scheme with two closed points that does not admit an affine open neighborhood and such that the semi-localization at the two points does not exist. See Appendix A.

Application: Ferrand pushouts. As an application of Theorem 1.2 we can prove that Ferrand pushouts [Fer03, TT16] exist for algebraic spaces and algebraic stacks. In the affine case, these are Milnor squares [Mil71, §2] and it follows that these are pushouts in the category of quasi-separated algebraic stacks.

Theorem 1.8 (Existence of Ferrand pushouts)**.** *Consider a diagram*

$$\begin{array}{ccc} \mathscr{X}_0 & \overset{i}{\hookrightarrow} & \mathscr{X} \\ {\scriptstyle f}\downarrow & & \\ \mathscr{Y}_0 & & \end{array}$$

of quasi-separated algebraic stacks where i is a closed immersion and f is affine. Then the pushout $\mathscr{Y}$ exists in the category of quasi-separated algebraic stacks and is a geometric pushout. If $\mathscr{X}_0$, $\mathscr{Y}_0$ and $\mathscr{X}$ are Deligne–Mumford stacks (resp. algebraic spaces, resp. affine schemes), then so is $\mathscr{Y}$.

Theorem 1.8 generalizes the main theorem of [TT16], where certain pushouts of algebraic spaces are proven to exist.

Application: Nisnevich neighborhoods. The following application is used in [HK19] and is a simple consequence of the local structure at non-closed points (Theorem 1.4).

Theorem 1.9 (Nisnevich neighborhoods of stacks with nice stabilizers)**.** *Let $\mathscr{X}$ be a quasi-compact and quasi-separated algebraic stack such that every, not necessarily closed, point of $\mathscr{X}$ has nice stabilizer group. Then there is a Nisnevich covering $f\colon \mathscr{W} \to \mathscr{X}$, where $\mathscr{W}$ is nicely fundamental. That is,*

(1) *f is étale and for every, not necessarily closed, point $x \in |\mathscr{X}|$ the restriction $f|_{\mathscr{G}_x}$ has a section.*

(2) *$\mathscr{W}$ admits an affine good moduli space W and there is a nice embeddable group scheme $G \to W$ such that $\mathscr{W} = [\operatorname{Spec} A/G]$.*

If $\mathscr{X}$ has affine (resp. separated) diagonal, then we can arrange that f is affine (resp. representable).

Remark 1.10. When $\mathscr{X}$ is an algebraic stack with a good moduli space such that every point of characteristic zero has an open neighborhood of characteristic zero, then $\mathscr{X}$ has a strong Nisnevich neighborhood of the form $[\operatorname{Spec} A/G]$ with G linearly reductive [AHR19, Thm. 13.1]. Here strong means that the Nisnevich neighborhood is a pull-back from a Nisnevich cover of the good moduli space. Note that the condition that $\mathscr{X}$ admits a good moduli space implies that every *closed* point has linearly reductive stabilizer.

In the case of linearly reductive stabilizers at *closed* points, we have the following result.

Theorem 1.11 (Nisnevich neighborhoods of stacks with linearly reductive stabilizers at closed points). *Let $\mathscr{X}$ be a quasi-compact and quasi-separated algebraic stack with affine stabilizers and linearly reductive stabilizers at closed points. Assume that $\mathscr{X}$ has separated (resp. quasi-affine, resp. affine) diagonal. Then there is a Nisnevich covering $f\colon [V/\mathrm{GL}_m] \to \mathscr{X}$, where V is a quasi-compact separated algebraic space (resp. quasi-affine scheme, resp. affine scheme). In general, the morphism f is not representable but if $\mathscr{X}$ has affine diagonal we can also arrange so that f is affine.*

When $\mathscr{X}$ has affine diagonal, the Nisnevich covering is fundamental but not always linearly fundamental. If $\mathscr{X}$ is the stack quotient of the non-separated affine line by $\mathbb{Z}/2\mathbb{Z} \times \mathbb{G}_m$ [AHR20, Ex. 5.2], then the unique closed point has stabilizer $\mathbb{G}_m$ whereas the open point has stabilizer $\mathbb{Z}/2\mathbb{Z}$. Every Nisnevich covering will thus have a point with stabilizer $\mathbb{Z}/2\mathbb{Z}$ and such stacks are not linearly fundamental in characteristic 2.

Application: Compact generation. Let $\mathscr{X}$ be a quasi-compact and quasi-separated algebraic stack and consider its unbounded derived category of $\mathcal{O}_{\mathscr{X}}$-modules with quasi-coherent cohomology sheaves $\mathrm{D}_{\mathsf{QCoh}}(\mathscr{X})$. A vexing question over the years has been whether the category $\mathrm{D}_{\mathsf{QCoh}}(\mathscr{X})$ is *compactly generated*. In this situation, this is equivalent to finding a set of perfect complexes $\{P_\lambda\}_{\lambda \in \Lambda}$ on $\mathscr{X}$ such that

(a) if $M \in \mathrm{D}_{\mathsf{QCoh}}(\mathscr{X})$ and $\mathrm{Hom}_{\mathcal{O}_{\mathscr{X}}}(P_\lambda, M) = 0$ for all $\lambda \in \Lambda$, then $M = 0$; and
(b) the functor $\mathrm{Hom}_{\mathscr{X}}(P_\lambda, -)\colon \mathrm{D}_{\mathsf{QCoh}}(\mathscr{X}) \to \mathsf{Ab}$ preserves small coproducts for all $\lambda \in \Lambda$.

For schemes, definitive positive results go back to the pioneering work of [TT90, Nee96]. For a thorough discussion on the subtleties of this question for algebraic stacks, we refer the interested reader to [HR17, HNR19].

A lot of progress was made on this question for stacks in [AHR20, Thm. 5.1] and [AHR19, Prop. 14.1], however. More precisely, [AHR20, Thm. 5.1] established compact generation provided that $\mathscr{X}$ had affine diagonal and the identity component G_x^0 of the stabilizer groups G_x of $\mathscr{X}$ at all points x of $\mathscr{X}$ were of multiplicative type. It was shown in [HNR19, Thm. 1.1], however, that if $\mathscr{X}$ had a point of positive characteristic y such that the reduced identity component $(G_y)_{\mathrm{red}}^0$ was not a torus, then $\mathrm{D}_{\mathsf{QCoh}}(\mathscr{X})$ was not compactly generated. In the following theorem we eliminate this discrepancy and give the following characterization of algebraic stacks in positive characteristic that have compactly generated derived categories.

Theorem 1.12. *Let $\mathscr{X}$ be a quasi-compact algebraic stack with affine diagonal satisfying* (PC). *The following conditions are equivalent.*

(1) $\mathscr{X}$ *is $\aleph_0$-crisp* [HR17, Defn. 8.1].
(2) $\mathrm{D}_{\mathsf{QCoh}}(\mathscr{X})$ *is compactly generated and for every closed subset $Z \subseteq |\mathscr{X}|$ with quasi-compact complement, there exists a perfect complex P on $\mathscr{X}$ with $\mathrm{supp}(P) = Z$.*
(3) $\mathrm{D}_{\mathsf{QCoh}}(\mathscr{X})$ *is compactly generated.*
(4) *For every point x of $\mathscr{X}$, the reduced identity component $(G_x)_{\mathrm{red}}^0$ of the stabilizer G_x at x is a torus.*
(5) *For every closed point x of $\mathscr{X}$, the reduced identity component $(G_x)_{\mathrm{red}}^0$ of the stabilizer G_x at x is a torus.*

We will prove Theorem 1.12 immediately after the non-noetherian local structure Theorem 5.1, and make use of the refinements established in [AHR19].

Application: Local structure theorem of derived algebraic stacks. We now come to the derived versions of our local structure results. Recall that a morphism f of derived stacks is *quasi-smooth* if f is locally of finite presentation and its cotangent complex $\mathbb{L}_f$ has Tor-amplitude ≤ 1. This is the analogue of lci maps in derived algebraic geometry.

Theorem 1.13 (Local structure of derived stacks). *Let $\mathscr{X}$ be a quasi-separated algebraic derived 1-stack with affine stabilizers. Let $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ be a closed substack and let $f_0 \colon \mathscr{W}_0 \to \mathscr{X}_0$ be a morphism with $(\mathscr{W}_0)_{\mathrm{cl}}$ linearly fundamental. Assume one of the following conditions:*

(1) *$\mathscr{W}_0$ satisfies (PC) or (N); or*
(2) *$\mathscr{X}_0$ satisfies (FC).*

Then

(a) *If f_0 is smooth (resp. étale), then there exists a smooth (resp. étale) morphism $f \colon \mathscr{W} \to \mathscr{X}$ such that $\mathscr{W}$ is fundamental and $f|_{\mathscr{X}_0} \cong f_0$.*
(b) *Assume that $(\mathscr{X}_0)_{\mathrm{cl}}$ has the resolution property. If f_0 is quasi-smooth then there exists a quasi-smooth morphism $f \colon \mathscr{W} \to \mathscr{X}$ such that $\mathscr{W}$ is fundamental and $f|_{\mathscr{X}_0} \cong f_0$ (here the restriction denotes the derived pull-back).*

It follows from Proposition 6.1 that $\mathscr{X}$ is linearly fundamental if and only if the underlying classical stack $\mathscr{X}_{\mathrm{cl}}$ is linearly fundamental. See Section 6 for further discussion.

Application: Local structure of a Θ-stratum. Let $\mathscr{S}$ be a quasi-separated algebraic stack and let $\mathscr{X}$ be an algebraic stack, quasi-separated and locally of finite presentation over $\mathscr{S}$ with affine stabilizers relative to $\mathscr{S}$. Let $\Theta := [\mathbb{A}^1/\mathbb{G}_m]$; then the mapping stack $\mathrm{Filt}(\mathscr{X}) := \underline{\mathrm{Map}}_{\mathscr{S}}(\Theta_{\mathscr{S}}, \mathscr{X})$ is also algebraic, locally of finite presentation, quasi-separated, and has affine stabilizers relative to $\mathscr{S}$ [HL14, Prop. 1.1.2]. A Θ-stratum in $\mathscr{X}$ is by definition an open and closed substack $\mathscr{Y} \subset \mathrm{Filt}(\mathscr{X})$ such that the morphism $\mathscr{Y} \to \mathscr{X}$ defined by restricting to $1 \in \Theta$ is a closed immersion, so that we may also regard $\mathscr{Y}$ as a closed substack of $\mathscr{X}$ (see [HL14, Defn. 2.1.1]).

Stratifications by closed substacks of this kind arise in geometric invariant theory, as well as on moduli stacks such as the moduli of torsion free sheaves on a projective scheme. In [AHLH18, Lem. 6.11], the following local structure result was established using our Theorem 5.1, and it is key to proving the semistable reduction theorem [AHLH18, Thm. 6.3].

Proposition 1.14. *Let S be a noetherian algebraic space. Let $\mathscr{X}$ be an algebraic stack of finite type over S with affine diagonal over S. If $\mathscr{Y} \hookrightarrow \mathscr{X}$ is a Θ-stratum, then there is a smooth representable morphism $p \colon [\mathrm{Spec}(A)/\mathbb{G}_m] \to \mathscr{X}$ such that $\mathscr{Y}$ is contained in the image of p, and $p^{-1}(\mathscr{Y})$ is the Θ-stratum*

$$p^{-1}(\mathscr{Y}) = [\mathrm{Spec}(A/I_+)/\mathbb{G}_m] \hookrightarrow [\mathrm{Spec}(A)/\mathbb{G}_m],$$

where $I_+ \subset A$ is the ideal generated by elements of positive degree.

2. Artin algebraization

In this section we prove Artin's algebraization theorem for linearly fundamental pairs (Theorem 2.3) which establishes Theorem 1.1 as a special case. In order to state the theorem, we will need the following terminology.
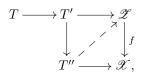
Definition 2.1. A *pair* $(\mathscr{X}, \mathscr{X}_0)$ consists of an algebraic stack $\mathscr{X}$ and a closed substack $\mathscr{X}_0$. We let $\mathcal{I}_{\mathscr{X}}$ denote the ideal defining $\mathscr{X}_0$ and let $\mathscr{X}_n$ denote the nth infinitesimal neighborhood of $\mathscr{X}_0$, that is, the closed substack defined by $\mathcal{I}_{\mathscr{X}}^{n+1}$. We

say that a pair $(\mathscr{X}, \mathscr{X}_0)$ has a given property $\mathcal{P}$ (e.g. linearly fundamental) if both $\mathscr{X}$ and $\mathscr{X}_0$ have $\mathcal{P}$.

A *morphism of pairs* $(\mathscr{X}, \mathscr{X}_0) \to (\mathscr{Y}, \mathscr{Y}_0)$ is a morphism $f: \mathscr{X} \to \mathscr{Y}$ such that $\mathscr{X}_0 \hookrightarrow f^{-1}(\mathscr{Y}_0)$, or equivalently, $f^{-1}\mathcal{I}_{\mathscr{Y}} \subseteq \mathcal{I}_{\mathscr{X}}$. For any $n \geq 0$, we let $f_n: \mathscr{X}_n \to \mathscr{Y}_n$ denote the induced morphism. We say that f is *adic* if $\mathscr{X}_0 = f^{-1}(\mathscr{Y}_0)$.

Note that if f is adic, then $\mathscr{X}_n = f^{-1}(\mathscr{Y}_n)$ for all n.

Definition 2.2. Let $f: \mathscr{Z} \to \mathscr{X}$ be a morphism of functors or stacks (e.g., schemes or algebraic spaces). Let T be a stack and $T \to \mathscr{Z}$ a morphism. We say that f is *formally versal at T* if the following condition holds: For every nilpotent immersions $T \hookrightarrow T' \hookrightarrow T''$ and 2-commutative diagram of solid arrows

$$
\begin{array}{ccc}
T \longrightarrow & T' \longrightarrow & \mathscr{Z} \\
\downarrow & \nearrow & \downarrow f \\
& T'' \longrightarrow & \mathscr{X},
\end{array}
$$

there exists a lift $T'' \to \mathscr{Z}$ and 2-morphisms that make the whole diagram 2-commutative.

Our main theorem is the following result, which generalizes [AHR20, Cor. A.19] and [AHR19, Thm. 12.14].

Theorem 2.3 (Algebraization of linearly fundamental pairs)**.** *Let S be an excellent affine scheme. Let $\mathscr{X}$ be an algebraic stack, locally of finite type over S with quasi-separated diagonal. Let $(\mathscr{Z}, \mathscr{Z}_0)$ be a complete linearly fundamental pair (Definition 2.5) over S such that $\mathscr{Z}_0$ is of finite type over S. Let $\eta: \mathscr{Z} \to \mathscr{X}$ be a morphism, formally versal at $\mathscr{Z}_0$. Then there exists*

(1) *a fundamental pair $(\mathscr{W}, \mathscr{W}_0)$ such that $\mathscr{W} \to S$ is of finite type and $\mathscr{W}_0$ is linearly fundamental;*

(2) *a morphism $\varphi: (\mathscr{Z}, \mathscr{Z}_0) \to (\mathscr{W}, \mathscr{W}_0)$ such that $\varphi_n: \mathscr{Z}_n \to \mathscr{W}_n$ is an isomorphism for all $n \geq 0$.*

(3) *a 2-commutative diagram over S*

$$
\mathscr{Z} \xrightarrow[\quad\eta\quad]{\;\;\varphi\;\;} \mathscr{W} \xrightarrow{\;\xi\;} \mathscr{X}
$$

In particular, the induced map $\widehat{\varphi}: \mathscr{Z} \to \widehat{\mathscr{W}}$ is an isomorphism and ξ is smooth in a neighborhood of $\mathscr{W}_0$.

Remark 2.4. Most of the statement of the theorem remains valid, with the same proof, when $\mathscr{X}$ is an arbitrary category fibered in groupoids that is locally of finite presentation over S. The only difference is that instead of a 2-isomorphism $\xi \circ \varphi \simeq \eta$, one only obtains a compatible family of 2-isomorphisms $\xi \circ \varphi|_{\mathscr{Z}_n} \simeq \eta|_{\mathscr{Z}_n}$ for all $n \geq 0$.

We prove this theorem at the end of the section after discussing some background material on pairs. We first explain how this theorem implies Theorem 1.1.

Proof of Theorem 1.1. Applying Theorem 2.3 and Remark 2.4 with $(\mathscr{Z}, \mathscr{Z}_0) := (Z, Z_0)$ gives a fundamental pair $(\mathscr{W}, \mathscr{W}_0)$ with $\mathscr{W}_0 \cong Z_0$. Since Z_0 is affine, we may apply [AHR19, Prop. 12.5] to the morphism $\mathscr{W} \to S$ to conclude that there is affine open neighborhood $U \subset \mathscr{W}$ of Z_0. Replacing $(\mathscr{W}, \mathscr{W}_0)$ with (U, Z_0) gives the result. $\square$

2.1. **Coherently complete pairs.** The following definition was introduced in [AHR20] and was further studied in [AHR19].

Definition 2.5. We say that a pair $(\mathscr{X}, \mathscr{X}_0)$ is *complete*, or that $\mathscr{X}$ is *coherently complete along* $\mathscr{X}_0$, if $\mathscr{X}$ is noetherian with affine diagonal and the induced functor $\mathsf{Coh}(\mathscr{X}) \to \varprojlim_n \mathsf{Coh}(\mathscr{X}_n)$ is an equivalence of abelian categories of coherent sheaves.

By Tannaka duality [HR19], we have that $\mathscr{X}$ is the colimit of $\{\mathscr{X}_n\}_{n \geq 0}$ in the category of noetherian stacks with quasi-affine diagonal and also in the category of noetherian stacks with affine stabilizers if $\mathscr{X}_0$ is quasi-excellent.

Let $(\mathscr{X}, \mathscr{X}_0)$ be a linearly fundamental noetherian pair. The good moduli space X is a noetherian affine scheme and $\pi\colon \mathscr{X} \to X$ is of finite type. This gives a morphism of pairs $(\mathscr{X}, \mathscr{X}_0) \to (X, X_0)$ where $X_0 = \pi(\mathscr{X}_0)$. The pair $(\mathscr{X}, \mathscr{X}_0)$ is complete if and only if (X, X_0) is complete [AHR19, Thm. 1.6]. The latter simply means that if $X = \operatorname{Spec} A$ and $X_0 = \operatorname{Spec} A/I$, then A is I-adically complete.

If $(\mathscr{X}, \mathscr{X}_0)$ is a fundamental noetherian pair such that $\mathscr{X}_0$ is linearly fundamental, then $(\widehat{\mathscr{X}}, \mathscr{X}_0)$ is a complete linearly fundamental pair where $\widehat{\mathscr{X}} = \mathscr{X} \times_X \widehat{X}$ and $\widehat{X} = \operatorname{Spec} \widehat{A}$ is the I-adic completion. Indeed, the completion factors through the Zariskification $\mathscr{X} \times_X \operatorname{Spec}\big((1+I)^{-1}A\big)$, which is linearly fundamental by [AHR19, Cor. 13.7].

2.2. **Preliminary results on pairs.** In this section, we provide criteria to check that a morphism of pairs is a closed immersion or isomorphism (Proposition 2.9) or is formally versal (Lemma 2.10).

Lemma 2.6. [Vas69, Prop. 1.2] *Let A be a ring and let $\phi\colon M \to N$ be a surjective homomorphism of finitely generated A-modules. If there exists an A-module isomorphism $M \cong N$, then ϕ is an isomorphism.*

Proof. We identify N with M and treat ϕ as an endomorphism of M. Then M is also a module over $A[t]$ where $tx = \phi(x)$ for $x \in M$. Since ϕ is surjective $tM = M$ and Nakayama's lemma tells us that there is an element $a \in A[t]$ such that $(1 - at)M = 0$. That is ϕ has inverse given by $\phi^{-1}(x) = ax$. $\qquad\square$

Lemma 2.7. *Suppose that $I \subseteq R$ is an ideal and $\varphi\colon R \to S$ is a surjective homomorphism of noetherian rings. If there is an abstract isomorphism of graded R/I-modules $\operatorname{Gr}_I R \to \operatorname{Gr}_{IS} S$ and I is nilpotent, then φ is an isomorphism.*

Proof. Since φ is surjective, it induces a surjection $\operatorname{Gr}_n \varphi\colon I^n/I^{n+1} \to I^n S/I^{n+1}S$ of finitely generated R/I-modules. By assumption, there is an abstract isomorphism $I^n/I^{n+1} \to I^n S/I^{n+1}S$ of R/I-modules, so $\operatorname{Gr}_n \varphi$ is an isomorphism by Lemma 2.6.

We have induced morphisms of exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I^d/I^{d+1} & \longrightarrow & R/I^{d+1} & \longrightarrow & R/I^d & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \operatorname{Gr}_d \varphi} & & \downarrow{\scriptstyle \varphi_{d+1}} & & \downarrow{\scriptstyle \varphi_d} & & \\
0 & \longrightarrow & I^d S/I^{d+1}S & \longrightarrow & S/I^{d+1}S & \longrightarrow & S/I^d S & \longrightarrow & 0
\end{array}
$$

and it follows that $\varphi_d\colon R/I^d \to S/I^d S$ is an isomorphism for every $d \geq 0$ by induction on d. Since $I^n = 0$ for sufficiently large n, the result follows. $\qquad\square$

The following results generalize [AHR20, Props. A.8 and A.10] from the local case.

Proposition 2.8. *Let $f\colon (\mathscr{X}, \mathscr{X}_0) \to (\mathscr{Y}, \mathscr{Y}_0)$ be a morphism of noetherian pairs.*

   (1) *If f_1 is a closed immersion, then so is f_n for every $n \geq 0$.*

(2) *If f_1 is a closed immersion and f_0 is an isomorphism, then f_n is adic for every $n \geq 0$.*
(3) *If f_1 is a closed immersion and there exists an isomorphism of graded $\mathcal{O}_{\mathscr{Y}_0}$-modules $\psi\colon \mathrm{Gr}_{\mathfrak{I}_{\mathscr{Y}}}(\mathcal{O}_{\mathscr{Y}}) \to (f_0)_* \mathrm{Gr}_{\mathfrak{I}_{\mathscr{X}}}(\mathcal{O}_{\mathscr{X}})$, then f_n is an isomorphism for every $n \geq 0$.*

Proof. We can replace f with f_n. The first part is then [AHR19, Lem. 6.3]: the question is local and reduces to the affine case where it follows from Nakayama's lemma. For the second part, we have seen that f_n is a closed immersion and then it is adic if and only if f_0 is an isomorphism. The third part is also local and thus follows from Lemma 2.7. $\square$

Proposition 2.9. *Let $f\colon (\mathscr{X}, \mathscr{X}_0) \to (\mathscr{Y}, \mathscr{Y}_0)$ be a morphism of complete pairs such that f_0 is an isomorphism.*

(1) *f is a closed immersion if and only if f_1 is a closed immersion.*
(2) *f is an isomorphism if and only if f_1 is a closed immersion and there exists an isomorphism $\psi\colon \mathrm{Gr}_{\mathfrak{I}_{\mathscr{Y}}}(\mathcal{O}_{\mathscr{Y}}) \to (f_0)_* \mathrm{Gr}_{\mathfrak{I}_{\mathscr{X}}}(\mathcal{O}_{\mathscr{X}})$ of graded $\mathcal{O}_{\mathscr{Y}_0}$-modules.*

Proof. The conditions are clearly necessary. Conversely, if the conditions of (1) (resp. (2)) hold, then f_n is adic and a closed immersion (resp. an isomorphism) for every $n \geq 0$ by Proposition 2.8. Since f_n is adic, we have that $f_n^{-1}(\mathscr{Y}_m) = \mathscr{X}_m$ for all $m \leq n$. Since $\mathscr{Y}$ is coherently complete along $\mathscr{Y}_0$, we obtain a closed substack $\mathscr{Z} \hookrightarrow \mathscr{Y}$ such that $\mathscr{Z} \times_{\mathscr{Y}} \mathscr{Y}_n = \mathscr{X}_n$ for all $n \geq 0$. Under condition (2), we have that $\mathscr{Z} = \mathscr{Y}$. Finally, since $(\mathscr{X}, \mathscr{X}_0)$ is complete, we have by Tannaka duality a unique isomorphism $\mathscr{X} \to \mathscr{Z}$ over $\mathscr{Y}$. $\square$

Let X be a quasi-compact and quasi-separated algebraic stack. Recall [HR17, Defn. 2.1] that X is said to have *cohomological dimension* 0 if $\mathrm{H}^i(X, M) = 0$ for all $i > 0$ and quasi-coherent $\mathcal{O}_X$-modules M. Affine schemes have cohomological dimension 0. More generally, cohomologically affine algebraic stacks that have affine diagonal or are noetherian and affine-pointed also have cohomological dimension 0 [HNR19, Thm. C.1].

Lemma 2.10. *Let $f\colon (\mathscr{Z}, \mathscr{Z}_0) \to (\mathscr{X}, \mathscr{X}_0)$ be a morphism of locally noetherian pairs. If $f_n\colon \mathscr{Z}_n \to \mathscr{X}_n$ is smooth for all $n \geq 0$, then f is formally versal at any morphism $T \to \mathscr{Z}$ from a quasi-compact and quasi-separated algebraic stack T of cohomological dimension 0 whose set theoretic image is contained in $|\mathscr{Z}_0|$.*

Proof. The lifting criterion in Definition 2.2 is equivalent to the same criterion for the map $f_n\colon \mathscr{Z}_n \to \mathscr{X}_n$ for $n \gg 0$ large enough that $\mathscr{Z}_n$ contains the image of T' and $\mathscr{X}_n$ contains the image of T'', so by our hypotheses we may assume that the map f is smooth. First note that T' has cohomological dimension 0 because any quasi-coherent $\mathcal{O}_{T'}$-module admits a finite filtration whose associated graded objects are pushforwards of objects in $\mathsf{QCoh}(T)$. Also, because we may factor $T' \to T''$ into a sequence of square-zero extensions, it suffices to verify the lifting criterion in the case where $T' \to T''$ is a square-zero extension by some $M \in \mathsf{QCoh}(T')$. In this case the obstruction to the existence of a dotted arrow is an element in the group $\mathrm{Ext}^1_{T'}(\mathbb{L}_{\mathscr{Z}/\mathscr{X}}|_{T'}, M)$. Since f is smooth, $\mathbb{L}_{\mathscr{Z}/\mathscr{X}}$ is a perfect complex of Tor-amplitude $[0, 1]$. Hence, the Ext group vanishes as T' has cohomological dimension 0. $\square$

2.3. **Proof of Theorem 2.3.** First we establish an important special case of Artin algebraization for pairs:

Lemma 2.11. *Let (S, S_0) be an excellent affine pair, let (T, T_0) be a complete affine pair, and let $f\colon (T, T_0) \to (S, S_0)$ be a morphism such that f_0 is an isomorphism*

and f_1 is a closed immersion. Let $\mathscr{X}$ be a finite type algebraic stack over S, and let $\mathscr{Z}_0 \hookrightarrow \mathscr{Z} := T \times_S \mathscr{X}$ be a closed substack over T_0. For any $N \geq 0$, there is an affine étale neighborhood $(S', S'_0) \to (S, S_0)$ and a closed substack $\mathscr{W} \hookrightarrow S' \times_S \mathscr{X}$ such that:

(1) The map $T \to S$ factors through S', and $T_N \to S'_N$ is a closed immersion;
(2) $T_N \times_T \mathscr{Z} = S'_N \times_{S'} \mathscr{W}$ as closed substacks of $S'_N \times_S \mathscr{X}$. In particular, if $\mathscr{W}_0 := \mathscr{Z}_0 \hookrightarrow \mathscr{W}$, then the canonical map is an isomorphism $\mathscr{Z}_N \cong \mathscr{W}_N$; and
(3) There is an isomorphism $\mathrm{Gr}_{\mathfrak{I}_{\mathscr{Z}}} \mathcal{O}_{\mathscr{Z}} \cong \mathrm{Gr}_{\mathfrak{I}_{\mathscr{W}}} \mathcal{O}_{\mathscr{W}}$ of graded modules over $\mathscr{Z}_0 \cong \mathscr{W}_0$.

Proof. Consider the functor $F \colon \mathsf{Sch}_{/S}^{\mathrm{op}} \to \mathsf{Set}$, where $F(U \to S)$ is the set of isomorphism classes of complexes of finitely presented quasi-coherent $\mathcal{O}_{U \times_S \mathscr{X}}$-modules $\mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{O}_{U \times_S \mathscr{X}}$ such that $\mathcal{E}_1$ is locally free. This functor is locally of finite presentation.

Let $\widehat{S}$ be the completion of S along S_0. Then $T \to \widehat{S}$ is a closed immersion by Proposition 2.9, because (T, T_0) is complete, f_0 is an isomorphism and f_1 is a closed immersion. Now let

$$\mathcal{O}_{\widehat{S}}^{\oplus n} \to \mathcal{O}_{\widehat{S}} \to \mathcal{O}_T$$

be a presentation of the structure sheaf of $T \hookrightarrow \widehat{S}$. Pulling back to $\widehat{S} \times_S \mathscr{X}$ we get a resolution

$$\ker(\alpha) \xrightarrow{\alpha} \mathcal{O}_{\widehat{S} \times_S \mathscr{X}}^{\oplus n} \xrightarrow{\beta} \mathcal{O}_{\widehat{S} \times_S \mathscr{X}} \twoheadrightarrow \mathcal{O}_{T \times_S \mathscr{X}}.$$

We regard the pair (α, β) as an element of $F(\widehat{S})$. Note that by increasing N if necessary, we may assume that both α and β satisfy the Artin–Rees condition $(\mathrm{AR})_N$ of [AHR20, Def. A.15] with respect to $\mathscr{Z}_0$.

Let (S^h, S_0) denote the henselization of the pair (S, S_0). By Artin approximation over henselian pairs [AHR19, Thm. 3.4] one can find a class in $F(S^h)$ which restricts to the same class as (α, β) in $F(S_N)$. Then because S^h is constructed as an inverse limit of étale neighborhoods of S_0, we lift this class in $F(S^h)$ to a class $(\alpha', \beta') \in F(S')$ for some étale map $S' \to S$ lying under S^h such that $S' \times_S S_0 \simeq S_0$.

We now let $\mathscr{W} \hookrightarrow S' \times_S \mathscr{X}$ be the closed substack defined by $\mathrm{im}(\beta') \subset \mathcal{O}_{S' \times_S \mathscr{X}}$. By construction we have

$$\mathcal{O}_{\mathscr{W}} \otimes_{\mathcal{O}_{S' \times_S \mathscr{X}}} \mathcal{O}_{S'_N} \simeq \mathrm{coker}(\beta'|_{S'_N}) \simeq \mathcal{O}_{T_N \times_S \mathscr{X}}$$

as $\mathcal{O}_{S' \times_S \mathscr{X}}$-algebras, which is the second condition of the lemma.

Now consider $(\alpha, \beta) \in F(\widehat{S})$ and the restriction of (α', β') to $F(\widehat{S})$. Both complexes are isomorphic after tensoring with $\mathcal{O}_{S_N}$, and by hypothesis the complex defined by (α, β) is exact and satisfies the Artin–Rees criterion $(\mathrm{AR})_N$, so the refined Artin–Rees theorem [AHR20, Thm. A.16] implies that

$$\mathrm{Gr}_{\mathfrak{I}_{\mathscr{Z}}} \mathcal{O}_{\mathscr{Z}} \cong \mathrm{Gr}_{\mathfrak{I}_{\mathscr{Z}}}(\mathrm{coker}(\beta)) \cong \mathrm{Gr}_{\mathfrak{I}_{\mathscr{W}}}(\mathrm{coker}(\beta')) \cong \mathrm{Gr}_{\mathfrak{I}_{\mathscr{W}}} \mathcal{O}_{\mathscr{W}}. \qquad \square$$

The following generalizes [AHR20, Thm. A.17].

Proposition 2.12 (Weak Artin algebraization for pairs)**.** *Let S be an excellent affine scheme, and let $\mathscr{X}$ be a category fibered in groupoids, locally of finite presentation over S. Let (T, T_0) be a noetherian affine pair over S such that $T_0 \to S$ is of finite type. Let $(\mathscr{Z}, \mathscr{Z}_0) \to (T, T_0)$ be a morphism of finite presentation and let $\eta \colon \mathscr{Z} \to \mathscr{X}$ be a morphism compatible over S. Fix an integer $N \geq 0$. Then there exists*

(1) *a pair $(\mathscr{W}, \mathscr{W}_0)$ of finite presentation over S, together with a morphism $\xi \colon \mathscr{W} \to \mathscr{X}$;*
(2) *an isomorphism $\mathscr{Z}_N \cong \mathscr{W}_N$ over $\mathscr{X}$; and*

(3) *an isomorphism* $\mathrm{Gr}_{\mathfrak{I}_{\mathscr{Z}}} \mathcal{O}_{\mathscr{Z}} \cong \mathrm{Gr}_{\mathfrak{I}_{\mathscr{W}}} \mathcal{O}_{\mathscr{W}}$ *of graded modules over* $\mathscr{Z}_0 \cong \mathscr{W}_0$.

Moreover, if $\mathscr{Z}$ *is fundamental, then one can arrange that* $\mathscr{W}$ *is fundamental.*

Proof. It suffices to prove the claims after base change to the completion of T, so we may assume that T is complete along T_0. Now write

$$T = \varprojlim_{\lambda} T_{\lambda},$$

where T_{λ} is a cofiltered system of affine S-schemes of finite type. For λ sufficiently large, $T_1 \to T_{\lambda}$ is a closed immersion. Increasing λ if necessary, standard limit methods give us an algebraic stack $\mathscr{Z}_{\lambda}$ of finite presentation over T_{λ} fitting into a commutative diagram

(1)
$$
\begin{array}{ccccc}
\mathscr{Z} & \rightrightarrows & \mathscr{Z}_{\lambda} & \rightrightarrows & \mathscr{X} \\
\downarrow & \square & \downarrow & & \downarrow \\
T & \longrightarrow & T_{\lambda} & \longrightarrow & S
\end{array}
$$

It now suffices to replace S with T_{λ}, and $\mathscr{X}$ with $\mathscr{Z}_{\lambda}$, and to find a stack over $\mathscr{Z}_{\lambda}$ meeting the conditions of the theorem. We may therefore assume that $\mathscr{X}$ is algebraic and of finite presentation over S, and that $T_1 \to S$ is a closed immersion, in which case the theorem follows immediately from Lemma 2.11 with S_0 as the image of T_0.

Finally, if $\mathscr{Z}$ were fundamental, meaning $\mathscr{Z}$ admits and affine map $f \colon \mathscr{Z} \to B\mathrm{GL}_{n,\mathbb{Z}}$ for some n, then in this case one can simultaneously approximate both the map f and the map $\mathscr{Z} \to \mathscr{X}$ by replacing $\mathscr{X}$ with $\mathscr{X} \times_S (B\mathrm{GL}_{n,S})$ in the argument above. The map $\mathscr{Z} \to B\mathrm{GL}_{n,S}$ is affine, so [Ryd15, Thm. C] guarantees that we can arrange for $\mathscr{Z}_{\lambda}$ in (1) to be affine over $B\mathrm{GL}_{n,S}$ as well. The stack $\mathscr{W}$ constructed in Lemma 2.11 will be affine over $B\mathrm{GL}_{n,S}$ as well, hence fundamental. $\square$

We now prove our main algebraization theorem:

Proof of Theorem 2.3. Let T be the good moduli space of $\mathscr{Z}$ and T_0 the good moduli space of $\mathscr{Z}_0$. Choose an $N \geq 1$. Then $T_0 \to S$ is of finite type, so Proposition 2.12 produces a stack $\mathscr{W}$ satisfying the first two conditions of the theorem along with a map $\xi \colon \mathscr{W} \to \mathscr{X}$ and an isomorphism $\psi_N \colon \mathscr{W}_N \cong \mathscr{Z}_N$ over $\mathscr{X}$.

We would like to extend the isomorphism ψ_N to a compatible sequence of isomorphisms $\psi_n \colon \mathscr{W}_n \to \mathscr{Z}_n$ over $\mathscr{X}$ for all $n \geq N$. Extending the map ψ_n to ψ_{n+1} is equivalent to finding a dotted arrow such that the diagram

$$
\begin{array}{ccc}
\mathscr{W}_n & \xrightarrow{\psi_n} & \mathscr{Z} \\
\downarrow {\scriptstyle \psi_{n+1}} \nearrow & & \downarrow {\scriptstyle \eta} \\
\mathscr{W}_{n+1} & \xrightarrow[\xi|_{\mathscr{W}_{n+1}}]{} & \mathscr{X}
\end{array}
$$

is 2-commutative. It is possible to do this for all $n \geq N$ because by hypothesis the map η is formally versal at $\mathscr{W}_0 = \mathscr{Z}_0$ (See Definition 2.2). The resulting sequence of maps $\psi_n \colon \mathscr{W}_n \to \mathscr{Z}_n$ and the induced map $\widehat{\psi} \colon \widehat{\mathscr{W}} \to \mathscr{Z}$ are isomorphisms by Proposition 2.9 and part (3) of Proposition 2.12. If we define φ to be the inverse of ψ followed by the canonical map $\widehat{\mathscr{W}} \to \mathscr{W}$, then by construction we have a compatible sequence of 2-isomorphisms $\xi \circ \varphi|_{\mathscr{Z}_n} \cong \eta|_{\mathscr{Z}_n}$ for all $n \geq 1$.

If $\mathscr{X}$ is an algebraic stack with quasi-separated diagonal, then the stack $I := \mathrm{Isom}_{\mathscr{X}}(\xi \circ \varphi, \eta)$ is a quasi-separated algebraic space, locally of finite type over $\mathscr{Z}$. The 2-isomorphisms $\xi \circ \varphi|_{\mathscr{Z}_n} \cong \eta|_{\mathscr{Z}_n}$ give a compatible sequence of sections σ_n of

$I \to \mathscr{Z}$ over $\mathscr{Z}_n$ for all $n \geq 1$. The image of all of the σ_n lie in some quasi-compact open substack $I' \subset I$, so we may replace I with I'. Then Tannaka duality implies that there is a unique section $\sigma \colon \mathscr{Z} \to I' \subset I$ of $I \to \mathscr{Z}$, which corresponds to a 2-isomorphism $\xi \circ \varphi \simeq \eta$ satisfying the conditions of the theorem. $\square$

3. Affine étale neighborhoods

In this section we prove the existence of affine étale neighborhoods (Theorem 1.2).

Proof of Theorem 1.2.

Step 1: Reduction to $\mathscr{X}$ of finite presentation over $\mathbb{Z}$. We may replace $\mathscr{X}$ with an open quasi-compact neighborhood of the image of W_0. Then $\mathscr{X}$ is quasi-compact and quasi-separated and hence of approximation type [Ryd16][1].

We can thus write $\mathscr{X}_0$ as the intersection of finitely presented closed immersions $\mathscr{X}_\lambda \hookrightarrow \mathscr{X}$ [Ryd15, Thm. D]. Using standard limit methods, we can thus, for sufficiently large λ, find an étale (resp. smooth) morphism $f_\lambda \colon W_0 \to \mathscr{X}_\lambda$ that restricts to $f_0 \colon W_0 \to \mathscr{X}_0$ [Ryd15, App. B]. After replacing f_0 with f_λ we can thus assume that $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ is of finite presentation.

Using [Ryd15, Thm. D] we can now write $\mathscr{X}$ as an inverse limit of stacks of finite presentation over $\operatorname{Spec}\mathbb{Z}$. Using standard limit methods, we can thus arrange so that the étale (resp. smooth) map $f_0 \colon W_0 \to \mathscr{X}_0$ and the closed immersion $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ arise as the pull-backs from stacks of finite presentation over $\operatorname{Spec}\mathbb{Z}$ [Ryd15, App. B].

In the two reduction steps above, we can also arrange so that W_0 remains affine by [Ryd15, Thm. C]. We can thus assume that $\mathscr{X}$ is of finite presentation over $\operatorname{Spec}\mathbb{Z}$.

Step 2: Existence of affine formal neighborhoods. Let $\mathscr{X}_n$ denote the nth infinitesimal neighborhood of $\mathscr{X}_0$ in $\mathscr{X}$. We claim that $f_0 \colon W_0 \to \mathscr{X}_0$ lifts to a compatible sequence of cartesian squares

$$
\begin{array}{ccc}
W_{n-1} & \longrightarrow & W_n \\
{\scriptstyle f_{n-1}}\downarrow & & \downarrow{\scriptstyle f_n} \\
\mathscr{X}_{n-1} & \longrightarrow & \mathscr{X}_n
\end{array}
$$

such that each f_n is étale (resp. smooth). Indeed, by [Ols06, Thm. 1.4], the obstruction to lifting f_{n-1} to f_n belongs to the group

$$
\operatorname{Ext}^2_{\mathcal{O}_{W_0}}\big(\mathbb{L}_{W_0/\mathscr{X}_0}, f_0^*(\mathcal{I}^n/\mathcal{I}^{n+1})\big),
$$

where $\mathcal{I}$ is the coherent ideal sheaf defining $\mathscr{X}_0 \hookrightarrow \mathscr{X}$. This group is zero since $\mathbb{L}_{W_0/\mathscr{X}_0} = \Omega_{W_0/\mathscr{X}_0}[0]$ is a vector bundle and W_0 is affine.

Since W_0 is affine, each W_n is also affine [Knu71, Cor. 3.6], [Ryd15, Cor. 8.2]. It follows from [EGA$_\mathrm{I}$, Cor. 0.7.2.8] that $Z := \operatorname{Spec}\big(\varprojlim_n \Gamma(W_n, \mathcal{O}_{W_n})\big)$ is a noetherian affine scheme complete along W_0 such that W_i is the ith infinitesimal neighborhood of W_0 in Z. By Tannaka duality [HR19], there is an induced morphism $\eta \colon Z \to \mathscr{X}$ which is formally versal at W_0 (Lemma 2.10). Note that Tannaka duality applies because we assume that $\mathscr{X}$ has affine stabilizers.

Step 3: Existence of étale neighborhoods. Applying Artin algebraization for pairs (Theorem 1.1) yields an affine scheme W of finite type over $\operatorname{Spec}\mathbb{Z}$, a closed immersion $W_0 \hookrightarrow W$, an isomorphism $\widehat{W} \to Z$, and a morphism $f \colon W \to \mathscr{X}$ extending $\eta|_{W_n}$ for all n; in particular, $f \colon W \to \mathscr{X}$ is étale (resp. smooth) along

---

[1]When f_0 is étale, we do not need [Ryd16]. Indeed, then $\mathscr{X}_0$ is Deligne–Mumford so after replacing $\mathscr{X}$ with an open neighborhood of $\mathscr{X}_0$, we may assume that $\mathscr{X}$ is Deligne–Mumford, hence of global type and approximation type [Ryd15, Def. 2.1, Prop. 2.10].

W_0. The preimage $f^{-1}(\mathscr{X}_0)$ is a closed subscheme of W which agrees with W_0 after restricting to the Zariski-localization of W along W_0. Therefore, there is an affine open subscheme $W' \subset W$ containing W_0 such that $f|_{W'}$ extends f_0. This finishes the proof of Theorem 1.2. $\qquad\square$

4. Existence of geometric pushouts

In this section, we prove Theorem 1.8, on the existence of pushouts of algebraic stacks. The exposition will follow [Hal17, App. A] closely, where a useful special case of this result was established. We begin with a definition.

Definition 4.1. Fix a 2-commutative square of algebraic stacks

$$
\begin{array}{ccc}
\mathscr{X}_0 & \overset{i}{\hookrightarrow} & \mathscr{X}_1 \\
f \downarrow & \Swarrow_\alpha & \downarrow f' \\
\mathscr{X}_2 & \underset{i'}{\hookrightarrow} & \mathscr{X}_3,
\end{array}
$$

where i and i' are closed immersions and f and f' are affine. If the induced map

$$
\mathcal{O}_{\mathscr{X}_3} \to i'_* \mathcal{O}_{\mathscr{X}_2} \times_{(i'f)_* \mathcal{O}_{\mathscr{X}_0}} f'_* \mathcal{O}_{\mathscr{X}_1}
$$

is an isomorphism of sheaves, then we say that the square is a *geometric pushout*, and that $\mathscr{X}_3$ is a *geometric pushout* of the diagram $[\mathscr{X}_2 \overset{f}{\leftarrow} \mathscr{X}_0 \overset{i}{\to} \mathscr{X}_1]$.

The main result of this section is the following refinement of Theorem 1.8. It also generalizes [Hal17, Prop. A.2] from the case of a locally nilpotent closed immersion to a general closed immersion.

Theorem 4.2. *Any diagram of algebraic stacks $[\mathscr{X}_2 \overset{f}{\leftarrow} \mathscr{X}_0 \overset{i}{\to} \mathscr{X}_1]$, where i is a closed immersion, f is affine, and $\mathscr{X}_1$ is quasi-separated, admits a geometric pushout $\mathscr{X}_3$. The resulting geometric pushout square is 2-cartesian and 2-cocartesian in the 2-category of algebraic stacks with quasi-separated diagonals. If $\mathscr{X}_1$ and $\mathscr{X}_2$ are quasi-compact (resp. quasi-separated, Deligne–Mumford, algebraic spaces, affine schemes), then so is $\mathscr{X}_3$.*

We will need the following two lemmas—the first is precisely [Hal17, Lem. A.3] and the second is a mild extension of [Hal17, Lem. A.4].
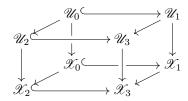
Lemma 4.3. *Fix a 2-commutative square of algebraic stacks*

$$
\begin{array}{ccc}
\mathscr{X}_0 & \overset{i}{\hookrightarrow} & \mathscr{X}_1 \\
f \downarrow & \Swarrow & \downarrow f' \\
\mathscr{X}_2 & \underset{i'}{\hookrightarrow} & \mathscr{X}_3.
\end{array}
$$

(1) *If the square is a geometric pushout, then it is 2-cartesian.*
(2) *If the square is a geometric pushout, then it remains so after flat base change on $\mathscr{X}_3$.*
(3) *If after fppf base change on $\mathscr{X}_3$ the square is a geometric pushout, then it was a geometric pushout prior to base change.*

Proof. The claim (1) is local on $\mathscr{X}_3$ for the smooth topology, thus we may assume that everything in sight is affine—whence the result follows from [Fer03, Thm. 2.2]. Claims (2) and (3) are trivial applications of flat descent. $\qquad\square$

Lemma 4.4. *Consider a 2-commutative diagram of algebraic stacks*

where the back and left faces of the cube are 2-cartesian and the top and bottom faces are geometric pushout squares. Then all faces of the cube are 2-cartesian. Moreover, if the morphisms $\mathscr{U}_1 \to \mathscr{X}_1$ and $\mathscr{U}_2 \to \mathscr{X}_2$ have one of the following properties:

(1) *flat;*
(2) *surjective;*
(3) *locally of finite type;*
(4) *flat and locally of finite presentation; or*
(5) *smooth.*

then the morphism $\mathscr{U}_3 \to \mathscr{X}_3$ has the same property.

Proof. By Lemma 4.3(2), this is all smooth local on $\mathscr{X}_3$ and $\mathscr{U}_3$; thus, we immediately reduce to the case where everything in sight is affine. Fix a diagram of rings $[A_2 \to A_0 \xleftarrow{p} A_1]$ where $p \colon A_1 \to A_0$ is surjective. For $j = 0, 1, 2$ fix A_j-algebras B_j and A_0-isomorphisms $B_2 \otimes_{A_2} A_0 \cong B_0$ and $B_1 \otimes_{A_1} A_0 \cong B_0$. Set $A_3 = A_2 \times_{A_0} A_1$ and $B_3 = B_2 \times_{B_0} B_1$; then we first have to prove that the natural maps $B_3 \otimes_{A_3} A_j \to B_j$ are isomorphisms, and that these isomorphisms are compatible with the given isomorphisms. This is an immediate consequence of [Fer03, Thm. 2.2(i)], since these are just questions about modules.

Case (1) similarly follows from [Fer03, Thm. 2.2(iv)]. Case (2) follows from the observation that $|\mathscr{X}_1| \amalg |\mathscr{X}_2| \to |\mathscr{X}_3|$ and $|\mathscr{U}_1| \amalg |\mathscr{U}_2| \to |\mathscr{U}_3|$ are surjective [Fer03, Sch. 4.3 & Thm. 5.1]. Case (5) follows from (4), the surjectivity of $|\mathscr{X}_1| \amalg |\mathscr{X}_2| \to |\mathscr{X}_3|$ already remarked, and the observation that smoothness is a fibral criterion for morphisms that are flat and locally of finite presentation.

For (3), we argue as follows: by [Fer03, Thm. 2.2(ii)], an A_3-module W_3 is zero if and only if the modules $W_3 \otimes_{A_3} A_1$ and $W_3 \otimes_{A_3} A_2$ are zero. Now write B_3 as the union of its finite type A_3-sub-algebras $B_{3,\lambda}$. As filtered direct limits commute with tensor products, it follows that for sufficiently large λ, the homomorphisms $B_{3,\lambda} \otimes_{A_3} A_1 \to B_1$ and $B_{3,\lambda} \otimes_{A_3} A_2 \to B_2$ are surjective. Looking at the cokernel, it follows that $B_{3,\lambda} \to B_3$ is surjective.

For (4): if B_j is a flat A_j-algebra of finite presentation for $j = 1, 2$, then we know by (3) that B_3 is of finite type. Hence, we can choose a surjection $P_3 = A_3[x_1, \ldots, x_n] \twoheadrightarrow B_3$. Let J_3 be the kernel. Since B_3 is A_3-flat, the sequence

$$0 \to J_3 \to P_3 \to B_3 \to 0$$

remains exact after tensoring by any A_3-algebra. In particular, $J_j = J_3 \otimes_{A_3} A_j$ is a $P_j = P_3 \otimes_{A_3} A_j$-module of finite type for $j = 1, 2$. It now follows from Ferrand's case of finite type modules (over the cocartesian square defined by the P_j) that J_3 is a P_3-module of finite type; hence B_3 is an A_3-algebra of finite presentation. $\square$
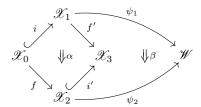
We now come to an important lemma, where we make use of Theorem 1.2 in a critical way. Note that the proof is almost identical to [Hal17, Lem. A.8].

Lemma 4.5. *Fix a 2-commutative square of algebraic stacks*

$$
\begin{array}{ccc}
\mathscr{X}_0 & \stackrel{i}{\hookrightarrow} & \mathscr{X}_1 \\
f \downarrow & \nearrow_\alpha & \downarrow f' \\
\mathscr{X}_2 & \stackrel{i'}{\hookrightarrow} & \mathscr{X}_3.
\end{array}
$$

If the square is a geometric pushout and i is a closed immersion, then the square is 2-cartesian and 2-cocartesian in the 2-category of algebraic stacks with quasi-separated diagonals.

Proof. That the square is 2-cartesian is Lemma 4.3(1). It remains to show that we can uniquely complete all 2-commutative diagrams of algebraic stacks

with a map $\mathscr{X}_3 \to \mathscr{W}$ and compatible 2-isomorphisms. By smooth descent, this is smooth-local on $\mathscr{X}_3$, so we may reduce to the situation where the $\mathscr{X}_j = \operatorname{Spec} A_j$ are all affine schemes. Since $\mathscr{X}_3$ is a geometric pushout of the diagram $[\mathscr{X}_2 \stackrel{f}{\leftarrow} \mathscr{X}_0 \stackrel{i}{\to} \mathscr{X}_1]$, it follows that $A_3 \cong A_2 \times_{A_0} A_1$.

Let $q \colon \operatorname{Spec} B \to \mathscr{W}$ be a smooth morphism such that the pullback $v_j \colon U_j \to \mathscr{X}_j$ of q along ψ_j is surjective for $j \in \{0, 1, 2\}$, which exists because the $\mathscr{X}_j$ are all quasi-compact. There are compatibly induced morphisms of quasi-separated algebraic spaces $\psi_{j,B} \colon U_j \to \operatorname{Spec} B$ for $j = 1$ and 2 and $f_B \colon U_0 \to U_2$ and $i_B \colon U_0 \hookrightarrow U_1$.

Let $c_2 \colon \operatorname{Spec} C_2 \to U_2$ be an étale morphism such that $v_2 \circ c_2$ is smooth and surjective. The morphism c_2 pulls back along f_B to give an étale morphism $c_0 \colon \operatorname{Spec} C_0 \to U_0$ such that $v_0 \circ c_0$ is smooth and surjective. Let $\tilde{f} \colon \operatorname{Spec} C_0 \to \operatorname{Spec} C_2$ and $\tilde{\psi}_2 \colon \operatorname{Spec} C_2 \to \operatorname{Spec} B$ be the resulting morphisms.

Since c_0 is étale and i is a closed immersion, it follows that there is an étale morphism $c_1 \colon \operatorname{Spec} C_1 \to U_1$ whose pullback along i_B is isomorphic to c_0 (Theorem 1.2). It can easily be arranged that $v_1 \circ c_1$ is smooth and surjective. Let $C_3 = C_2 \times_{C_0} C_1$. Then there is a uniquely induced ring homomorphism $A_3 \to C_3$. By Lemma 4.4, the morphism $c_3 \colon \operatorname{Spec} C_3 \to \operatorname{Spec} A_3$ is smooth and surjective. Hence, we may replace $\operatorname{Spec} A_j$ by $\operatorname{Spec} C_j$ and further assume that the ψ_j for $j = 0$, 1, and 2 factor through some smooth morphism $q \colon \operatorname{Spec} B \to \mathscr{W}$. In particular, there is an induced morphism $\psi_3 \colon \operatorname{Spec} A_3 \to \operatorname{Spec} B \to \mathscr{W}$.

It remains to prove that the morphism ψ_3 is unique up to a unique choice of 2-morphism. Let ψ_3 and $\psi_3' \colon \operatorname{Spec} A_3 \to \mathscr{W}$ be two compatible morphisms. That these morphisms are isomorphic can be checked smooth-locally on $\operatorname{Spec} A_3$. But smooth-locally, the morphisms ψ_3 and ψ_3' both factor through some $\operatorname{Spec} B \to \mathscr{W}$ and the morphisms $\operatorname{Spec} A_j \to \operatorname{Spec} A_3 \to \operatorname{Spec} B$ coincide for $j = 0$, 1, and 2, thus ψ_3 and ψ_3' are isomorphic. To show that the isomorphism between ψ_3 and ψ_3' is unique, we just repeat the argument, and the result follows. □

We finally come to the proof of Theorem 4.2.

Proof of Theorem 4.2. By Lemma 4.5, it suffices to prove the existence of geometric pushouts. Let $\mathscr{C}_0$ denote the category of affine schemes. For $d = 1$, 2, 3, let $\mathscr{C}_d$ denote the full 2-subcategory of the 2-category of algebraic stacks with affine dth

diagonal. Note that $\mathscr{C}_3$ is the full 2-category of algebraic stacks. We will prove by induction on $d \geq 0$ that if $[\mathscr{X}_2 \xleftarrow{f} \mathscr{X}_0 \xrightarrow{i} \mathscr{X}_1]$ belongs to $\mathscr{C}_d$ and $\mathscr{X}_1$ is quasi-separated, then it admits a geometric pushout. For the base case, where $d = 0$ and $\mathscr{X}_j = \operatorname{Spec} A_j$ is affine, take $\mathscr{X}_3 = \operatorname{Spec}(A_2 \times_{A_0} A_1)$ and the result is clear.

Now let $d > 0$ and assume that $[\mathscr{X}_2 \xleftarrow{f} \mathscr{X}_0 \xrightarrow{i} \mathscr{X}_1]$ belongs to $\mathscr{C}_d$. Fix a smooth surjection $\amalg_{l \in \Lambda} X_2^l \to \mathscr{X}_2$, where X_2^l is an affine scheme $\forall l \in \Lambda$. Set $X_0^l = X_2^l \times_{\mathscr{X}_2} \mathscr{X}_0$. As f is affine, the scheme X_0^l is also affine. By Theorem 1.2, the resulting smooth surjection $X_0^l \to \mathscr{X}_0$ lifts to a smooth surjection $X_1^l \to \mathscr{X}_1$, with X_1^l affine, and $X_0^l \cong X_1^l \times_{\mathscr{X}_1} \mathscr{X}_0$. For $j = 0, 1,$ and 2 and $u, v, w \in \Lambda$ set $X_j^{uv} = X_j^u \times_{\mathscr{X}_j} X_j^v$ and $X_j^{uvw} = X_j^u \times_{\mathscr{X}_j} X_j^v \times_{\mathscr{X}_j} X_j^w$. Note that for $j = 0, 1,$ and 2 and all $u, v, w \in \Lambda$ we have $X_j^{uv}, X_j^{uvw} \in \mathscr{C}_{d-1}$. By the inductive hypothesis, for $I = u, uv$ or uvw, a geometric pushout X_3^I of the diagram $[X_2^I \leftarrow X_0^I \to X_1^I]$ exists. By Lemma 4.5, there are uniquely induced morphisms $X_j^{uv} \to X_j^u$. For $j \neq 3$, these morphisms are clearly smooth, and by Lemma 4.4 the morphisms $X_3^{uv} \to X_3^u$ are smooth. It easily verified that the universal properties give rise to a smooth groupoid $[\amalg_{u,v \in \Lambda} X_3^{uv} \rightrightarrows \amalg_{w \in \Lambda} X_3^w]$. The quotient $\mathscr{X}_3$ of this groupoid in the category of stacks is algebraic. By Lemma 4.3(3) it is also a geometric pushout of the diagram $[\mathscr{X}_2 \leftarrow \mathscr{X}_0 \to \mathscr{X}_1]$ and the result follows.

That the pushout inherits the properties "quasi-compact" and "quasi-separated" follows from $\mathscr{X}_1 \amalg \mathscr{X}_2 \to \mathscr{X}_3$ being affine and surjective. The properties "Deligne–Mumford" and "algebraic space", are inherited since $(\mathscr{X}_1 \setminus \mathscr{X}_0) \amalg \mathscr{X}_2 \to \mathscr{X}_3$ is a surjective monomorphism. For "affine", this was the base case of the induction. $\square$

5. LOCAL STRUCTURE OF ALGEBRAIC STACKS

In this section we prove the main local structure results for stacks (Theorems 1.3 and 1.4) as well as non-noetherian generalizations (Theorems 5.1 and 5.8).

5.1. **Proof of Theorem 1.3.** When $f_0 \colon \mathscr{W}_0 \to \mathscr{X}_0$ is smooth or étale, the theorem can be established along similar lines to [AHR19, Proof of Thm. 12.1].

Proof of Theorem 1.3(1)—smooth/étale case.

Step 1: An effective formally versal solution. Since $\mathscr{W}_0$ is quasi-compact, we may assume that $\mathscr{X}$ is quasi-compact after replacing $\mathscr{X}$ with a quasi-compact open substack containing the image of $\mathscr{W}_0$. Since $\mathscr{W}_0$ is linearly fundamental, we can apply [AHR19, Thm. 1.11] to obtain a cartesian square

$$
\begin{array}{ccc}
\mathscr{W}_0 & \hookrightarrow & \widehat{\mathscr{W}} \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \widehat{f}} \\
\mathscr{X}_0 & \hookrightarrow & \mathscr{X}
\end{array}
$$

where $\widehat{f} \colon \widehat{\mathscr{W}} \to \mathscr{X}$ is a flat morphism and $\widehat{\mathscr{W}}$ is a linearly fundamental stack coherently complete along $\mathscr{W}_0$. Since $\mathscr{W}_n \to \mathscr{X}_n$ is smooth, $\widehat{f} \colon \widehat{\mathscr{W}} \to \mathscr{X}$ is formally versal at $\mathscr{W}_0$ (Lemma 2.10).

Step 2: Algebraization. We now apply algebraization for linearly fundamental pairs (Theorem 2.3) to the pair $(\widehat{\mathscr{W}}, \mathscr{W}_0)$ and morphism $\widehat{f} \colon \widehat{\mathscr{W}} \to \mathscr{X}$ to obtain a fundamental pair $(\mathscr{W}, \mathscr{W}_0)$ with $\mathscr{W}$ of finite type over S and a morphism $f \colon \mathscr{W} \to \mathscr{X}$ smooth (resp. étale) along $\mathscr{W}_0$ such that $\widehat{\mathscr{W}}$ is isomorphic over $\mathscr{X}$ to the coherent completion of $\mathscr{W}$ along $\mathscr{W}_0$. After replacing $\mathscr{W}$ with an open neighborhood, we may arrange that f is smooth (resp. étale) and $\mathscr{W}$ is fundamental. Indeed, if $\mathscr{U} \subset \mathscr{W}$ is an open neighborhood of $\mathscr{W}_0$ such that $f|_{\mathscr{U}}$ is smooth (resp. étale) and if $\pi \colon \mathscr{W} \to W$

denotes the adequate moduli space, then we replace $\mathscr{W}$ with the inverse image of any affine open subscheme of $W \smallsetminus \pi(\mathscr{W} \smallsetminus \mathscr{U}))$ containing $\pi(\mathscr{W}_0)$. $\qquad\square$
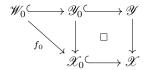
The case when $f_0\colon \mathscr{W}_0 \to \mathscr{X}_0$ is syntomic is handled by reducing to the smooth case.

Proof of Theorem 1.3(2)—syntomic case.

Step 1: We may assume that $f_0\colon \mathscr{W}_0 \to \mathscr{X}_0$ is affine. We may assume that $\mathscr{X}$ is quasi-compact. Since $\mathscr{W}_0$ is fundamental, there is an affine morphism $\mathscr{W}_0 \to B\mathrm{GL}_n$ for some n. Since $\mathscr{X}_0$ has affine diagonal (as it has the resolution property), the induced morphism $\mathscr{W}_0 \to \mathscr{X}_0 \times B\mathrm{GL}_n$ is affine. Since $B\mathrm{GL}_n$ is smooth with smooth diagonal, we may replace $(\mathscr{X}, \mathscr{X}_0)$ with $(\mathscr{X} \times B\mathrm{GL}_n, \mathscr{X}_0 \times B\mathrm{GL}_n)$.

Step 2: There is a factorization $f_0\colon \mathscr{W}_0 \hookrightarrow \mathscr{Y}_0 \to \mathscr{X}_0$ where $\mathscr{W}_0 \hookrightarrow \mathscr{Y}_0$ is a regular closed immersion, $\mathscr{Y}_0 \to \mathscr{X}_0$ is smooth and affine, and $\mathscr{Y}_0$ is linearly fundamental. Since $\mathscr{X}_0$ has the resolution property and $(f_0)_*\mathcal{O}_{\mathscr{W}_0}$ is a finite type $\mathcal{O}_{\mathscr{X}_0}$-algebra, there exists a vector bundle $\mathcal{E}_0$ on $\mathscr{X}_0$ and a surjection $\mathrm{Sym}(\mathcal{E}_0) \twoheadrightarrow (f_0)_*\mathcal{O}_{\mathscr{W}_0}$. Setting $\mathscr{Y}_0 = \mathbb{V}(\mathcal{E}_0) = \mathrm{Spec}(\mathrm{Sym}\,\mathcal{E}_0)$ yields a factorization such that $\mathscr{W}_0 \hookrightarrow \mathscr{Y}_0$ is regular closed immersion and $\mathscr{Y}_0 \to \mathscr{X}_0$ is smooth and affine. To arrange that $\mathscr{Y}_0$ is linearly fundamental, we apply the étale case of the local structure theorem (Theorem 1.3(1)) to the closed immersion $\mathscr{W}_0 \hookrightarrow \mathscr{Y}_0$ to extend the isomorphism $\mathscr{W}_0 \xrightarrow{\sim} \mathscr{W}_0$ to an étale morphism $\mathscr{Y}_0' \to \mathscr{Y}_0$ with $\mathscr{Y}_0'$ fundamental. Since $\mathscr{W}_0$ satisfies (PC) or (N), or $\mathscr{Y}_0$ satisfies (FC), there is an open neighborhood of $\mathscr{W}_0 \hookrightarrow \mathscr{Y}_0'$ that is linearly fundamental [AHR19, Prop. 16.14].

Step 3: Apply the smooth version of the local structure theorem. Since $\mathscr{Y}_0$ is linearly fundamental, we may apply the smooth case of the local structure theorem (Theorem 1.3(1)) to the closed immersion $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ and smooth morphism $\mathscr{Y}_0 \to \mathscr{X}_0$ to obtain a commutative diagram

$$
\begin{array}{ccccc}
\mathscr{W}_0 & \!\!\lhook\joinrel\longrightarrow\!\! & \mathscr{Y}_0 & \!\!\lhook\joinrel\longrightarrow\!\! & \mathscr{Y} \\
& {\scriptstyle f_0}\searrow & \downarrow & \square & \downarrow \\
& & \mathscr{X}_0 & \!\!\lhook\joinrel\longrightarrow\!\! & \mathscr{X}
\end{array}
$$

with a cartesian square such that $\mathscr{Y} \to \mathscr{X}$ is smooth and $\mathscr{Y}$ is fundamental.

Step 4: Lift the closed substack $\mathscr{W}_0 \hookrightarrow \mathscr{Y}_0$ to a closed substack $\mathscr{W} \hookrightarrow \mathscr{Y}$ syntomic over $\mathscr{X}$. Let $\mathcal{I}_0$ be the ideal sheaf defining $\mathscr{W}_0 \hookrightarrow \mathscr{Y}_0$ and consider the conormal bundle $\mathcal{N}_0 := \mathcal{I}_0/\mathcal{I}_0^2$. After replacing $\mathscr{Y}$ with an étale fundamental neighborhood of $\mathscr{W}_0 \hookrightarrow \mathscr{Y}$, we may extend the conormal bundle to a vector bundle $\mathcal{N}$ on $\mathscr{Y}$; this follows from applying [AHR19, Prop. 16.12] to the fundamental pair $(\mathscr{Y}, \mathscr{W}_0)$ and the morphism $\mathscr{W}_0 \to B\mathrm{GL}_n$ induced from $\mathcal{N}_0$.

We claim that after replacing $\mathscr{Y}$ with a fundamental étale neighborhood of $\mathscr{W}_0$ the canonical homomorphism $\mathcal{N} \twoheadrightarrow \mathcal{N}_0 \hookrightarrow \mathcal{O}_{\mathscr{Y}_0}/\mathcal{I}_0^2$ extends to a diagram

$$
\begin{array}{ccc}
\mathcal{N} & \dashrightarrow & \mathcal{O}_{\mathscr{Y}} \\
\downarrow & & \downarrow \\
\mathcal{N}_0 & \!\!\lhook\joinrel\longrightarrow\!\! & \mathcal{O}_{\mathscr{Y}_0}/\mathcal{I}_0^2.
\end{array}
$$

If $\mathscr{Y}$ is linearly fundamental, this is immediate as the functor $\mathrm{Hom}_{\mathcal{O}_{\mathscr{Y}}}(\mathcal{N}, -) = \Gamma(\mathscr{Y}, \mathcal{N}^\vee \otimes -)$ is exact. In general, let $\mathscr{W}_1 \hookrightarrow \mathscr{Y}_0$ be the closed substack defined by $\mathcal{I}_0^2$. Then we have a morphism $\mathscr{W}_1 \to \mathbb{V}(\mathcal{N})$ over $\mathscr{Y}$ which by [AHR19, Prop. 16.12] extends to a section $\mathscr{Y} \to \mathbb{V}(\mathcal{N})$ after replacing $\mathscr{Y}$ with a fundamental étale neighborhood of $\mathscr{W}_0$. This gives the requested map $\mathcal{N} \to \mathcal{O}_{\mathscr{Y}}$.

Let $\mathscr{W}$ be the closed substack defined by the image of $\mathcal{N} \to \mathcal{O}_{\mathscr{Y}}$. By construction $\mathscr{W}$ contains the closed substack $\mathscr{W}_0$. We claim that $\mathscr{W}_0 \hookrightarrow \mathscr{W} \times_{\mathscr{X}} \mathscr{X}_0$ is an isomorphism and $\mathscr{W} \to \mathscr{X}$ is syntomic in an open neighborhood of $\mathscr{W}_0$. This establishes the theorem as we may shrink further to arrange that $\mathscr{W} \to \mathscr{X}$ is syntomic in a fundamental open neighborhood of $\mathscr{W}_0$. These claims can be verified smooth-locally on $\mathscr{X}$ and $\mathscr{Y}$ so we may assume that $\mathscr{X}$ and $\mathscr{Y}$ are affine schemes and $\mathcal{N}$ is a trivial vector bundle. By construction $\mathcal{N}\mathcal{O}_{\mathscr{Y}_0} + \mathcal{J}_0^2 = \mathcal{J}_0$ so it follows that $\mathscr{W}_0 \hookrightarrow \mathscr{W} \times_{\mathscr{Y}} \mathscr{Y}_0$ is an isomorphism in an open neighborhood of $\mathscr{W}_0$ by Nakayama's lemma.

Let $f_1, \ldots, f_n \in \mathcal{O}_{\mathscr{Y}}$ be the image of a basis of $\mathcal{N}$. We claim that $f_1, \ldots, f_n$ is a regular sequence in a neighborhood of $\mathscr{W}_0$ and that $\mathscr{W} \to \mathscr{X}$ is flat in a neighborhood of $\mathscr{W}_0$. By [EGA$_{\mathrm{IV}}$, Thm. 11.3.8 (c) $\implies$ (b')], it is enough to prove that the images of $f_1, \ldots, f_n$ in $\mathcal{O}_{\mathscr{Y}_x, w}$ is a regular sequence for every $w \in |\mathscr{W}_0|$ with image $x \in |\mathscr{X}_0|$, which follows by construction. $\qquad\square$

5.2. **Non-noetherian local structure theorem.** The following provides a non-noetherian generalization of Theorem 1.3, which we establish by reducing to the noetherian case.

Theorem 5.1 (Local structure of stacks). *Let $\mathscr{X}$ be a quasi-separated algebraic stack with affine stabilizers, $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ be a closed substack and $f_0 \colon \mathscr{W}_0 \to \mathscr{X}_0$ be a morphism with $\mathscr{W}_0$ linearly fundamental. Assume one of the following conditions:*

  (1) *$\mathscr{X}$ is locally of finite type over an excellent algebraic space; or*
  (2) *$\mathscr{W}_0$ satisfies (PC) or (N); or*
  (3) *$\mathscr{X}_0$ satisfies (FC).*

Then

  (a) *If f_0 is smooth (resp. étale), then there exists a smooth (resp. étale) morphism $f \colon \mathscr{W} \to \mathscr{X}$ such that $\mathscr{W}$ is fundamental and $f|_{\mathscr{X}_0} \simeq f_0$.*
  (b) *Assume that $\mathscr{W}_0$ satisfies (PC) or (N) or $\mathscr{X}_0$ satisfies (FC). If f_0 is syntomic and $\mathscr{X}_0$ has the resolution property, then there exists a syntomic morphism $f \colon \mathscr{W} \to \mathscr{X}$ such that $\mathscr{W}$ is fundamental and $f|_{\mathscr{X}_0} \cong f_0$.*

Proof. Case (1) is precisely Theorem 1.3. For cases (2)–(3), after replacing $\mathscr{X}$ with a quasi-compact open substack we can assume that $\mathscr{X}$ is quasi-compact. If $\mathscr{X}_0$ satisfies (FC), we can assume that $\mathscr{X}$ is also (FC). Indeed, let S be the spectrum of $\mathbb{Z}$ localized in the characteristics of $\mathscr{X}_0$ and replace $\mathscr{X}$, $\mathscr{X}_0$ and $\mathscr{W}_0$ with their base changes along $S \to \operatorname{Spec} \mathbb{Z}$. Once the theorem is established in this case, we can use standard limit methods to replace S with an open subscheme of $\operatorname{Spec} \mathbb{Z}$. If instead $\mathscr{X}_0$ satisfies (PC) or (N), we let $S = \operatorname{Spec} \mathbb{Z}$.

By [Ryd16], we can write $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ as a limit of finitely presented closed immersions $\mathscr{X}_{0,\lambda} \hookrightarrow \mathscr{X}$ with transition maps that are closed immersions. For sufficiently large λ, we can extend $f_0 \colon \mathscr{W}_0 \to \mathscr{X}_0$ to a map $f_{0,\lambda} \colon \mathscr{W}_{0,\lambda} \to \mathscr{X}_{0,\lambda}$ of finite presentation. For sufficiently large λ, we have that $f_{0,\lambda}$ is smooth/étale/syntomic. If $\mathscr{X}_0$ has the resolution property, then so does $\mathscr{X}_{0,\lambda}$ for sufficiently large λ. This follows from the Totaro–Gross characterization of the resolution property as having a quasi-affine morphism to $B\mathrm{GL}_N$ for some N [Tot04, Gro17] and [Ryd15, Thm. C]. For sufficiently large λ, we also have that $\mathscr{W}_{0,\lambda}$ is linearly fundamental [AHR19, Thm. 15.3] using that either $\mathscr{W}_0$ is (PC) or (N), or $\mathscr{X}$ is (FC). After replacing $\mathscr{W}_0$ and $\mathscr{X}_0$ with $\mathscr{W}_{0,\lambda}$ and $\mathscr{X}_{0,\lambda}$ we may thus assume that $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ is of finite presentation.

Using [Ryd16], we may further write $\mathscr{X} \to S$ as a limit of algebraic stacks $\mathscr{X}_\lambda \to S$ of finite presentation. For sufficiently large λ, we can descend the finitely presented maps $f_0 \colon \mathscr{W}_0 \to \mathscr{X}_0$ and $i \colon \mathscr{X}_0 \hookrightarrow \mathscr{X}$ to finitely presented maps $f_{0,\lambda} \colon \mathscr{W}_{0,\lambda} \to \mathscr{X}_{0,\lambda}$ and $i_\lambda \colon \mathscr{X}_{0,\lambda} \hookrightarrow \mathscr{X}_\lambda$. For sufficiently large λ, we have that $\mathscr{X}_\lambda$ has affine stabilizers [HR15, Thm. 2.8] and, as before, that $f_{0,\lambda}$ is smooth/étale/syntomic, that

i_λ is a closed immersion, that $\mathscr{W}_{0,\lambda}$ is linearly fundamental and that $\mathscr{X}_{0,\lambda}$ has the resolution property. We are now in the situation of Theorem 1.3. $\square$

5.3. Compact generation.
We can now prove Theorem 1.12 on the compact generation of algebraic stacks in positive characteristic.

Proof of Theorem 1.12. The implications $(1) \implies (2) \implies (3)$ and $(4) \implies (5)$ are trivial. The implication $(3) \implies (5)$ is [HNR19, Thm. 1.1] since every closed point has positive characteristic. It remains to prove that (5) implies (1) and (4). To this end, let x be a closed point of $\mathscr{X}$, which we view as morphism $x \colon \operatorname{Spec} l \to \mathscr{X}$, where l is an algebraically closed field. Let $i_x \colon \mathcal{G}_x \hookrightarrow \mathscr{X}$ be the closed immersion of the residual gerbe of x. Then there is a field $\kappa(x)$ such that $\mathcal{G}_x \to \operatorname{Spec} \kappa(x)$ is a coarse moduli space. Certainly, $\kappa(x) \subseteq l$. After taking a finite extension $\kappa(x) \subseteq k \subseteq l$, $(\mathcal{G}_x)_k \simeq BH$, for some group scheme H over k. After passing to an additional finite extension of k, there is a subgroup scheme $H' \hookrightarrow H$ such that $H'_l \simeq G^0_{\mathrm{red}}$. By assumption, G^0_{red} is a torus, so H' is of multiplicative type. Set $\mathscr{W}^x_0 = BH'$, $\mathscr{X}^x_0 = \mathcal{G}_x$ and let $f^x_0 \colon \mathscr{W}^x_0 \to \mathscr{X}^x_0$ be the induced morphism. We claim that f^x_0 is syntomic. Indeed, f^x_0 is the composition $BH' \to BH \to \mathcal{G}_x$. Now $BH \to \mathcal{G}_x$ is the base change of $\operatorname{Spec} k \to \operatorname{Spec} \kappa(x)$, which is syntomic. Also, $BH' \to BH$ is fppf-locally the morphism $H/H' \to \operatorname{Spec} k$. Since $H \to \operatorname{Spec} k$ is syntomic (Lemma 5.2) and $H \to H/H'$ is fppf, $H/H' \to \operatorname{Spec} k$ is syntomic. By descent, $BH' \to BH$ is syntomic and so f^x_0 is too.

We now apply Theorem 5.1(2)(b) to f^x_0: this results in a syntomic morphism $f^x \colon \mathscr{W}^x \to \mathscr{X}$ such that $\mathscr{W}^x$ is fundamental and $f^x|_{\mathscr{X}^x_0} \simeq f^x_0$. Since $\mathscr{W}^x$ is fundamental and f^x_0 is finite, we may shrink $\mathscr{W}^x$ so that f^x is quasi-finite [AHR20, Lem. 3.1]. Additionally, since $\mathscr{X}$ has affine diagonal, we may further shrink $\mathscr{W}^x$ so that f^x is affine [AHR19, Prop. 12.5]. By [AHR19, Prop. 13.4], after passing to a strictly étale neighborhood of $\mathscr{W}^x_0$, we may further shrink $\mathscr{W}^x$ so that it is nicely fundamental.

Since $\mathscr{X}$ is quasi-compact and the f^x are all open morphisms, there is a finite set of closed points x_1, ..., x_m of $\mathscr{X}$ such that the induced morphism $f \colon \mathscr{W} = \amalg^m_{i=1} \mathscr{W}^{x_i} \xrightarrow{\amalg f^{x_i}} \mathscr{X}$ is affine, quasi-finite, syntomic, and faithfully flat. But $\mathscr{W}$ is nicely fundamental, so it is $\aleph_0$-crisp [HR17, Ex. 8.6]. By [HR17, Thm. C], $\mathscr{W}$ is $\aleph_0$-crisp. This proves $(5) \implies (1)$. Since $\mathscr{W} \to \mathscr{X}$ is quasi-finite and surjective and the reduced identity components of the stabilizers of $\mathscr{W}$ are tori, so are those of $\mathscr{X}$. This proves $(5) \implies (4)$. $\square$

We include the following result below for lack of reference.

Lemma 5.2. *Let S be an algebraic space. If $G \to S$ is a group algebraic space that is flat and locally of finite presentation, then it is syntomic.*

Proof. Since $G \to S$ is flat and locally of finite presentation, we reduce immediately to the situation where S is the spectrum of an algebraically closed field k [SP, Tags 01UF & 069N]. Let $G^0 \subseteq G$ be the connected component of the identity, which is a normal, irreducible, and quasi-compact flat closed subgroup scheme of G [SP, Tag 0B7R]. Since G is locally of finite type, $G^0 \subseteq G$ is even open and closed. Hence, the quotient G/G^0 is étale. Thus, we may replace G with G^0 and assume that G is connected and of finite type. If the characteristic of k is 0, then $G \to S$ is smooth and we are done (Cartier's Theorem [SP, Tag 047N]). In general, there is an extension of groups $1 \to G_{\mathrm{ant}} \to G \to G_{\mathrm{aff}} \to 1$, where G_{ant} is anti-affine (i.e., $\Gamma(G_{\mathrm{ant}}, \mathcal{O}_{G_{\mathrm{ant}}}) \simeq k$) and G_{aff} is affine [Bri09, (0.2)]. Then G_{ant} is smooth, so it suffices to prove the claim when G is affine. In this case, $G \hookrightarrow \mathrm{GL}_n$ for some $n > 0$. Then GL_n/G is smooth and so the morphism $\mathrm{GL}_n \to \mathrm{GL}_n/G$ is syntomic; hence, $G \to \operatorname{Spec} k$ is syntomic. $\square$

5.4. **Local structure of stacks at pro-affine-immersions.** We recall [TT17, §3]: a morphism of algebraic stacks $j\colon \mathscr{U} \to \mathscr{X}$ is a *pro-open immersion* if every morphism $\mathscr{Y} \to \mathscr{X}$ with set-theoretic image contained in $|j(\mathscr{U})|$ factors uniquely through j. It is established in [TT17, Prop. 3.1.4] that j is necessarily a flat monomorphism and $|j(\mathscr{U})| = \cap_{\mathscr{V} \supseteq |j(\mathscr{U})|} \mathscr{V}$, where the intersection ranges over all open stacks $\mathscr{V} \subseteq \mathscr{X}$ containing $j(\mathscr{U})$. If j is quasi-compact, then it is a pro-open immersion if and only if it is a flat monomorphism [TT17, Thm. 3.2.5] and then j is quasi-affine [Ray68, Prop. 1.5 (ii)]. If j is quasi-compact, then it is also a topological embedding [Ray68, Prop. 1.2] and if in addition $\mathscr{X}$ is quasi-compact, then $|j(\mathscr{U})| = \cap_{\mathscr{V} \supseteq |j(\mathscr{U})|} \mathscr{V}$, where the intersection ranges over the quasi-compact opens of $\mathscr{X}$ containing $j(\mathscr{U})$.

Remark 5.3. Let $j\colon \mathscr{U} \to \mathscr{X}$ be a quasi-compact pro-open immersion of algebraic stacks. There is a factorization of j as $\mathscr{U} \xrightarrow{j'} \mathscr{X}' \xrightarrow{g} \mathscr{X}$, where j' is an affine pro-open immersion and g is a quasi-compact open immersion [Ray68, Prop. 1.5 (i)].

We introduce the following variant: a morphism of algebraic stacks $j\colon \mathscr{U} \to \mathscr{X}$ is a *pro-affine(-open) immersion* if $\mathscr{U}$ represents a cofiltered intersection $\cap_\alpha \mathscr{V}_\alpha$, where the $\mathscr{V}_\alpha \subseteq \mathscr{X}$ are (open) immersions and the transition maps $\mathscr{V}_\alpha \to \mathscr{V}_\beta$, which are automatically (open) immersions, are eventually affine.

Example 5.4. An immersion of algebraic stacks is a pro-affine-immersion.

Example 5.5. If $x \in |\mathscr{X}|$ is a point of a quasi-separated algebraic stack, then the inclusion $\mathscr{G}_x \to \mathscr{X}$ of the residual gerbe is a pro-affine-immersion [HR18, Lem. 2.1].

Remark 5.6. A pro-affine-open immersion of algebraic stacks is pro-étale.

Remark 5.7. If $\mathscr{X}$ is a normal and $\mathbb{Q}$-factorial noetherian stack, then any quasi-compact pro-open immersion $j\colon \mathscr{U} \to \mathscr{X}$ is pro-affine-open. This follows from the result [Ray68, Cor. 2.7]: after restricting to an open substack, the complement of $\mathscr{U}$ is a, possibly infinite, union of Cartier divisors and the complements of finite unions of these divisors are affine open immersions.

The following theorem simultaneously generalizes Theorem 1.4 and Theorem 5.1. Note that in Theorem 1.4 no extra conditions are needed as (FC) always holds for the residual gerbe as it is a one-point space.

Theorem 5.8 (Local structure of stacks at pro-affine-immersions). *Assumptions and conclusions as in Theorem 5.1 (2) or (3) except that $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ is a pro-affine-immersion.*

Proof. As a first preliminary step, we can as before assume that $\mathscr{X}$ is quasi-compact and, if $\mathscr{X}_0$ satisfies (FC), that $\mathscr{X}$ satisfies (FC) by base changing along $S \to \operatorname{Spec} \mathbb{Z}$ where S is the spectrum of $\mathbb{Z}$ localized in the characteristics of $\mathscr{X}_0$.

By assumption, $\mathscr{X}_0 = \cap_\lambda \mathscr{X}_\lambda$ is an intersection of a cofiltered system of immersions $\mathscr{X}_\lambda \hookrightarrow \mathscr{X}$ with eventually affine inclusions $\mathscr{X}_\mu \hookrightarrow \mathscr{X}_\lambda$. Pick α sufficiently large such that $\mathscr{X}_\lambda \hookrightarrow \mathscr{X}_\alpha$ is affine for all $\lambda \geq \alpha$ and pick a quasi-compact open neighborhood $\mathscr{U}$ of $\mathscr{X}_0$ in $\mathscr{X}_\alpha$. Then $\mathscr{X}_0 = \cap_{\lambda \geq \alpha}(\mathscr{X}_\lambda \cap \mathscr{U})$ so we may assume that all the $\mathscr{X}_\lambda$ are quasi-compact and that all the $\mathscr{X}_\mu \hookrightarrow \mathscr{X}_\lambda$ are affine.

By standard limit methods, the morphism $f_0\colon \mathscr{W}_0 \to \mathscr{X}_0$ descends to a morphism $f_\alpha\colon \mathscr{W}_\alpha \to \mathscr{X}_\alpha$, which is étale, smooth or syntomic if f_0 is so. If $\lambda \geq \alpha$, set $\mathscr{W}_\lambda = \mathscr{W}_\alpha \times_{\mathscr{X}_\alpha} \mathscr{X}_\lambda$. Then $\mathscr{W}_0 = \cap_{\lambda \geq \alpha} \mathscr{W}_\lambda$. Now either $\mathscr{X}$ satisfies (FC) (by the initial reduction) or $\mathscr{W}_0$ satisfies (PC) or (N). Hence, $\mathscr{W}_\beta$ is linearly fundamental for some $\beta \gg \alpha$ [AHR19, Prop. 15.3]. After replacing $\mathscr{X}$ with an open neighborhood of $\mathscr{X}_\beta$, we may assume that $\mathscr{X}_\beta \hookrightarrow \mathscr{X}$ is a closed immersion. We may now apply Theorem 5.1 (2) or (3) to f_β and the result follows. $\square$

We now prove the refinements.

Proof of Theorem 1.5. Arguing as in the proof of Theorem 5.8, we may assume that $\mathscr{W}_0$ satisfies (PC) or (N) or $\mathscr{W}$ satisfies (FC). We may further assume that there is a factorization of $\mathscr{W}_0 \hookrightarrow \mathscr{W} \to \mathscr{X}$ through an immersion $\mathscr{W}_\beta \hookrightarrow \mathscr{W}$ such that $\mathscr{W}_\beta$ is fundamental and $\mathscr{W}_\beta \hookrightarrow \mathscr{W} \to \mathscr{X}$ is representable [Ryd15, Thm. C]. We now factor $\mathscr{W}_\beta \hookrightarrow \mathscr{W}$ as $\mathscr{W}_\beta \hookrightarrow \mathscr{Z} \subseteq \mathscr{W}$, where $\mathscr{W}_\beta \hookrightarrow \mathscr{Z}$ is a closed immersion and $\mathscr{Z} \subseteq \mathscr{W}$ is an open immersion. In this generality, however, $\mathscr{Z}$ is not necessarily fundamental (it can be arranged to be if $\mathscr{W}_0 \hookrightarrow \mathscr{W}$ is a closed immersion, however). But we can now apply Theorem 5.1 to the closed immersion $\mathscr{W}_\beta \hookrightarrow \mathscr{Z}$. We thus obtain an étale neighborhood $p\colon \mathscr{W}' \to \mathscr{Z}$ of $\mathscr{W}_\beta$ such that $\mathscr{W}'$ is fundamental and the induced morphism $\mathscr{W}'_\beta = p^{-1}(\mathscr{W}_\beta) \simeq \mathscr{W}_\beta \to \mathscr{Z} \subseteq \mathscr{W} \to \mathscr{X}$ is representable. The result now follows from [AHR19, Prop. 12.5] applied to the pair $(\mathscr{W}', \mathscr{W}'_\beta)$ and the morphism $\mathscr{W}' \to \mathscr{X}$. $\qquad\square$

Proof of Theorem 1.6. As in the proof of Theorem 1.6, we may assume that there is a factorization of $\mathscr{W}_0 \hookrightarrow \mathscr{W}$ through an immersion $\mathscr{W}_\beta \hookrightarrow \mathscr{W}$ such that $\mathscr{W}_\beta$ is linearly or nicely fundamental, respectively. Likewise, if $\mathscr{W}_0 = [\operatorname{Spec} A_0/G_0]$, then we can arrange so that $\mathscr{W}_\beta = [\operatorname{Spec} A_\beta/G_\beta]$ with G_β linearly reductive or nice.

We have a closed then open factorization $\mathscr{W}_\beta \hookrightarrow \mathscr{Z} \subseteq \mathscr{W}$. Apply Theorem 5.1 to $\mathscr{W}_\beta \hookrightarrow \mathscr{Z}$ to replace $\mathscr{W}$ with an étale neighborhood of $\mathscr{W}_\beta$ that is fundamental. We can now apply [AHR19, Props. 16.11, 16.12 and 16.14] and the result follows. $\quad\square$

5.5. **Nisnevich neighborhoods.**

Proof of Theorem 1.9. We apply Theorem 1.4 to *every* point of $|\mathscr{X}|$: for each $x \in |\mathscr{X}|$ we obtain an étale morphism $f_x\colon \mathscr{W}_x \to \mathscr{X}$ such that $f_x|_{\mathscr{G}_x}$ is an isomorphism and $\mathscr{W}_x$ is fundamental. If $\mathscr{X}$ has affine (resp. separated) diagonal, then Theorem 1.5 says that we can arrange that f_x is affine (resp. representable). By Theorem 1.6, we may further assume that $\mathscr{W}_x$ is nicely fundamental. Set $\mathscr{W} = \amalg_{x \in |\mathscr{X}|} \mathscr{W}_x$ and take $f = \amalg_x f_x\colon \mathscr{W} \to \mathscr{X}$; then f is a quasi-separated Nisnevich covering. By [HR18, Prop. 3.3], we may shrink $\mathscr{W}$ so that it is quasi-compact (a monomorphic splitting sequence must factor through finitely many of the $\mathscr{W}_x$), remains nicely fundamental and f is a Nisnevich covering. $\qquad\square$

Proof of Theorem 1.11. We apply Theorem 5.1 to every closed point of $|\mathscr{X}|$: for each closed point x of $\mathscr{X}$ we obtain an étale morphism $g\colon \mathscr{W} \to \mathscr{X}$ such that $\mathscr{W} = [U/\operatorname{GL}_n]$ is fundamental and $g|_{\mathscr{G}_x}$ is an isomorphism. If $\mathscr{X}$ has affine (resp. separated diagonal), then Theorem 1.5 says that we can arrange that g is affine (resp. representable).

For an integer $d \geq 1$, let $\mathscr{W}^d$ be the dth fiber product of g; then the symmetric group S_d acts on $\mathscr{W}^d$ by permuting the factors. Let e be the maximum rank of a fiber of g. Then there is an induced Nisnevich covering $f\colon \coprod_{1 \leq d \leq e}[\mathscr{W}^d/S_d] \to \mathscr{X}$ since g is representable.

Let V^d be the dth fiber product of $U \to \mathscr{W} \to \mathscr{X}$. Then $\mathscr{W}^d = [V^d/(\operatorname{GL}_n)^d]$. Let P be one of the properties: separated, quasi-affine, affine. If the diagonal of $\mathscr{X}$ has property P, then the algebraic space V^d has property P. Since the Stiefel manifold $\operatorname{GL}_{dn}/(\operatorname{GL}_n)^d$ is affine, it follows that $\mathscr{W}^d = [V'/\operatorname{GL}_{nd}]$ for an algebraic space V' with property P.

Let $p\colon \mathscr{W}^d \to [\mathscr{W}^d/S_d]$. Let $\mathcal{E}$ be the vector bundle on $\mathscr{W}^d$ with frame bundle V'. Then we claim that the frame bundle V of $p_*\mathcal{E}$ is an algebraic space with property P. Indeed, V is an algebraic space since the stabilizers of $[\mathscr{W}^d/S_d]$ act faithfully on $p_*\mathcal{E}$, cf. [EHKV01, Lem. 2.13]. Since $p^*V \to V$ is finite, étale and surjective, it is enough to prove that p^*V has property P. But since p is finite étale, we have that $p^*p_*\mathcal{E} \to \mathcal{E}$ is split surjective and it follows that p^*V has property P by

considering Stiefel manifolds again. We have thus shown that $[\mathscr{W}^d/S_d] = [V/\mathrm{GL}_N]$ for an algebraic space V with property P.

When $\mathscr{X}$ has affine diagonal, then $\mathscr{W}^d \to \mathscr{X}$ is affine but $[\mathscr{W}^d/S_d] \to \mathscr{X}$ is merely separated. Let $\mathrm{SEC}^d(\mathscr{W}/\mathscr{X}) \subseteq \mathscr{W}^d$ be the open and closed substack that is the complement of all diagonals. Then S_d acts freely on $\mathrm{SEC}^d(\mathscr{W}/\mathscr{X})$ relative to $\mathscr{X}$ and $\mathrm{\acute{E}T}^d(\mathscr{W}/\mathscr{X}) := [\mathrm{SEC}^d(\mathscr{W}/\mathscr{X})/S_d] \to \mathscr{X}$ is affine and an étale neighborhood of any point of $\mathscr{X}$ at which g has rank d. Thus $f\colon \coprod_{1 \leq d \leq e} \mathrm{\acute{E}T}^d(\mathscr{W}/\mathscr{X}) \to \mathscr{X}$ is a fundamental Nisnevich covering with f affine. $\qquad\square$

5.6. **Existence of henselizations.**

Proof of Theorem 1.7. By Theorem 5.8, there exists an étale neighborhood $\mathscr{W} \to \mathscr{X}$ of $\mathscr{W}_0 := \mathscr{X}_0$ such that $\mathscr{W}$ is fundamental. Let $\mathscr{W}_0 \to W_0$ and $\mathscr{W} \to W$ be the good and adequate moduli spaces. We claim that the henselization W^h of W along W_0 exists and is affine. If $\mathscr{W}_0$ is a closed substack, then this follows from [Ray70, Ch. XI, Thm. 2] as W is affine. If $\mathscr{W}_0 = \mathscr{G}_x$ is the residual gerbe of a point $x \in |\mathscr{X}|$, then $W_0 = \mathrm{Spec}\,\kappa(x) \hookrightarrow W$ is the inclusion of a point w and $W^h = \mathrm{Spec}\,\mathcal{O}_{W,w}^h$. In this case, we also note that $\mathscr{W}_0$ satisfies (FC). Let $\mathscr{W}^h = \mathscr{W} \times_W W^h$. Since $W^h \to W$ is flat, $\mathscr{W}^h \to W^h$ is an adequate moduli space. By [AHR19, Thm. 3.6], $(\mathscr{W}^h, \mathscr{W}_0)$ is a henselian pair and by [AHR19, Thm. 13.7], $\mathscr{W}^h$ is linearly fundamental since the closed points of $\mathscr{W}^h$ have linearly reductive stabilizer. To show that $\mathscr{W}^h \to \mathscr{X}$ is the henselization of $\mathscr{X}$ along $\nu\colon \mathscr{X}_0 \hookrightarrow \mathscr{X}$, it is enough to prove that any quasi-separated étale neighborhood $g\colon \mathscr{W}' \to \mathscr{W}^h$ of $\mathscr{W}_0$ has a section. This is precisely the conclusion of [AHR19, Prop. 16.4]. $\qquad\square$

6. Local structure of derived algebraic stacks

In this section we give a derived version of the local structure theorem.

An algebraic derived 1-stack is the derived analogue of an algebraic stack: it is a sheaf of ∞-groupoids on the opposite of the ∞-category of simplicial commutative rings (with its étale topology) that admits a surjective morphism, represented by smooth derived algebraic spaces, from a disjoint union of derived affine schemes.

Let $\mathscr{X}$ be an algebraic derived 1-stack. We say that $\mathscr{X}$ is *fundamental* if there exists an affine morphism $\mathscr{X} \to B\mathrm{GL}_n$; that is, if $\mathscr{X} = [\mathrm{Spec}\,A/\mathrm{GL}_n]$ for some derived affine scheme A. We say that $\mathscr{X}$ is *linearly fundamental* if it is fundamental and cohomologically affine, that is, $R\Gamma(\mathscr{X}, -)$ is t-exact.

Proposition 6.1 (Derived effectivity theorem). *Let $\mathscr{X}_0 \hookrightarrow \mathscr{X}_1 \hookrightarrow \mathscr{X}_2 \hookrightarrow \dots$ be a sequence of derived thickenings, i.e., $\mathscr{X}_m \cong \tau_{\leq m}\mathscr{X}_n$ for every $m \leq n$. If $\mathscr{X}_0$ is linearly fundamental, then there is a linearly fundamental algebraic derived 1-stack $\mathscr{X}$ and a compatible sequence of equivalences $\tau_{\leq n}\mathscr{X} \cong \mathscr{X}_n$.*

Proof. The existence and uniqueness of an algebraic 1-stack with compatible isomorphisms $\tau_{\leq n}\mathscr{X} \cong \mathscr{X}_n$ is given by [Lur04, Prop. 5.4.6]. $\mathscr{X}$ is determined by the equivalence $\mathscr{X}(A) \cong \mathscr{X}_n(A)$ for n-truncated simplicial commutative rings, and $\mathscr{X}(A) \cong \varprojlim_n \mathscr{X}(\tau_{\leq n}A)$ in general.

Let $f_0\colon \mathscr{X}_0 \to B\mathrm{GL}_r$ be an affine morphism. The obstruction to lifting a morphism $f_n\colon \mathscr{X}_n \to B\mathrm{GL}_r$ to $f_{n+1}\colon \mathscr{X}_{n+1} \to B\mathrm{GL}_r$ lies in

$$\mathrm{Ext}^1_{\mathscr{X}_0}\big(f_0^*\mathbb{L}_{B\mathrm{GL}_r}, \pi_{n+1}(\mathcal{O}_{\mathscr{X}_{n+1}})[n+1]\big).$$

This obstruction group vanishes since $\mathscr{X}_0$ is linearly fundamental. We can thus find a compatible sequence of morphisms $f_n\colon \mathscr{X}_n \to B\mathrm{GL}_r$. Since f_0 is affine, so is f_n for every n. The compatible family of morphisms $f_n\colon \mathscr{X}_n \to B\mathrm{GL}_r$ defines a morphism $f\colon \mathscr{X} \to B\mathrm{GL}_r$, and the resulting morphism is affine because it is affine on every truncation.

Finally, because pushforward along the inclusion $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ is t-exact and identifies $\mathsf{QCoh}(\mathscr{X})^\heartsuit \cong \mathsf{QCoh}(\mathscr{X}_0)^\heartsuit$, $\mathscr{X}$ is cohomologically affine if and only if $R\Gamma(\mathscr{X}_0, -)$ has cohomological dimension 0 on $\mathsf{QCoh}(\mathscr{X}_0)^\heartsuit$. Since $\mathscr{X}_0$ has affine diagonal, this is the same as being cohomologically affine. □

Proof of Theorem 1.13. If $\mathscr{X}_0$ satisfies (FC), then let S be the spectrum of $\mathbb{Z}$ localized in the characteristics of $\mathscr{X}_0$ and base change everything along $S \to \mathrm{Spec}\,\mathbb{Z}$. At the very end, we can then replace S by an open quasi-compact subscheme of $\mathbb{Z}$.

First assume that f_0 is smooth. Then $\mathscr{W}_0 \times_{\mathscr{X}_0} (\mathscr{X}_0)_{\mathrm{cl}}$ is classical and we may apply the classical version of the local structure theorem (Theorem 5.1). This gives us a fundamental classical stack $\mathscr{W}_{\mathrm{cl}}$ and a smooth morphism $f_{\mathrm{cl}} \colon \mathscr{W}_{\mathrm{cl}} \to \mathscr{X}_{\mathrm{cl}}$. Since either (PC)/(N) holds for $\mathscr{W}_0$ or (FC) for $\mathscr{W}_{\mathrm{cl}}$, we may assume that $\mathscr{W}_{\mathrm{cl}}$ is linearly fundamental (Theorem 1.6). We may now deform $f_{\leq 0} := f_{\mathrm{cl}}$ to smooth maps $f_{\leq n} \colon \mathscr{W}_{\leq n} \to \tau_{\leq n}\mathscr{X}$ for every n. Indeed, the obstruction lies in

$$\mathrm{Ext}^2_{\mathscr{W}_{\mathrm{cl}}}(\mathbb{L}_{f_{\mathrm{cl}}}, f^*_{\mathrm{cl}}\pi_n(\mathcal{O}_{\mathscr{X}})[n]),$$

which vanishes as $\mathscr{W}_{\mathrm{cl}}$ is cohomologically affine and f_{cl} is smooth. By Proposition 6.1, there is a linearly fundamental derived 1-stack $\mathscr{W}$ with compatible isomorphisms $\mathscr{W}_{\leq n} \cong \tau_{\leq n}\mathscr{W}$. Because both $\mathscr{W}$ and $\mathscr{X}$ are nilcomplete [Lur04, Prop. 5.3.7], the smooth morphisms $\mathscr{W}_{\leq n} \to \tau_{\leq n}\mathscr{X}$ extend uniquely to a smooth morphism $f \colon \mathscr{W} \to \mathscr{X}$. Since

$$\mathrm{Ext}^1_{(\mathscr{W}_0)_{\mathrm{cl}}}(\mathbb{L}_{(f_0)_{\mathrm{cl}}}, (f_0)^*_{\mathrm{cl}}\pi_n(\mathcal{O}_{\mathscr{X}_0})[n]) = 0,$$

the isomorphism $\mathscr{W}_0 \times_{\mathscr{X}_0} (\mathscr{X}_0)_{\mathrm{cl}} \to \mathscr{W} \times_{\mathscr{X}} (\mathscr{X}_0)_{\mathrm{cl}}$ extends to an isomorphism $\mathscr{W}_0 \to \mathscr{W} \times_{\mathscr{X}} \mathscr{X}_0$ over $\mathscr{X}_0$.

When instead f_0 is quasi-smooth, we proceed as in the syntomic case of the classical version of the local structure theorem, see the proof of Theorem 1.3(2).

Step 1: First replace $\mathscr{X}$ with $\mathscr{X} \times B\mathrm{GL}_n$ so that $\mathscr{W}_0 \to \mathscr{X}_0$ becomes affine.

Step 2: Consider the morphism of classical stacks $(\mathscr{W}_0)_{\mathrm{cl}} \to (\mathscr{X}_0)_{\mathrm{cl}}$ and pick a factorization $(\mathscr{W}_0)_{\mathrm{cl}} \to \mathscr{Y}_0 \to (\mathscr{X}_0)_{\mathrm{cl}}$ where the first map is a closed immersion and the second map is affine and smooth. Here we use that $(\mathscr{X}_0)_{\mathrm{cl}}$ has the resolution property. Then apply the classical étale version of the structure theorem to $(\mathscr{W}_0)_{\mathrm{cl}} = (\mathscr{W}_0)_{\mathrm{cl}} \hookrightarrow \mathscr{Y}_0$. We can thus replace $\mathscr{Y}_0$ with an étale neighborhood of $(\mathscr{W}_0)_{\mathrm{cl}}$ and assume that $\mathscr{Y}_0$ is linearly fundamental.

Step 3: Apply the smooth case of the derived local structure theorem to $\mathscr{Y}_0 \to (\mathscr{X}_0)_{\mathrm{cl}} \hookrightarrow \mathscr{X}$ and we obtain a smooth map $\mathscr{Y} \to \mathscr{X}$. Since $\mathscr{Y} \to \mathscr{X}$ is smooth and $\mathscr{W}_0$ is linearly fundamental, the obstructions to lifting the closed immersion $(\mathscr{W}_0)_{\mathrm{cl}} \hookrightarrow \mathscr{Y}$ to closed immersions $\tau_{\leq n}(\mathscr{W}_0) \hookrightarrow \mathscr{Y}$ over $\mathscr{X}$ for every n vanish. We obtain a closed immersion $\mathscr{W}_0 \hookrightarrow \mathscr{Y}$ because both $\mathscr{W}_0$ and $\mathscr{Y}$ are nilcomplete [Lur04, Prop. 5.3.7]. Since either (PC)/(N) holds for $\mathscr{W}_0$ or (FC) for $\mathscr{Y}$, we may assume that $\mathscr{Y}$ is linearly fundamental (Theorem 1.6).

Step 4: Now let $\mathscr{Y}_0 = \mathscr{Y} \times_{\mathscr{X}} \mathscr{X}_0$ (previously it denoted its classical truncation). The morphism $\mathscr{W}_0 \to \mathscr{Y}_0$ is a quasi-smooth closed immersion. Let $\mathcal{N} = \pi_1(\mathbb{L}_{\mathscr{W}_0/\mathscr{Y}_0})$ denote the corresponding conormal bundle on $(\mathscr{W}_0)_{\mathrm{cl}}$. After replacing $\mathscr{Y}$ with an étale neighborhood of $\mathscr{W}_0$ [AHR19, Prop. 16.12], we may assume that $\mathcal{N}$ extends to a vector bundle $\mathcal{E}$ on $\mathscr{Y}$.

Let F denote the homotopy fiber of $\mathcal{O}_{\mathscr{Y}_0} \to \mathcal{O}_{\mathscr{W}_0}$. Since the Hurewicz map $F \otimes_{\mathcal{O}_{\mathscr{Y}_0}} \mathcal{O}_{\mathscr{W}_0} \to \mathbb{L}_{\mathscr{W}_0/\mathscr{Y}_0}[-1]$ is an isomorphism on π_0, we have an induced isomorphism $\mathcal{E}|_{(\mathscr{W}_0)_{\mathrm{cl}}} \simeq \mathcal{N} \simeq F|_{(\mathscr{W}_0)_{\mathrm{cl}}}$. Since $\mathscr{Y}_0$ is cohomologically affine, this lifts to a map $\mathcal{E}|_{\mathscr{Y}_0} \to F$. The composition $s_0 \colon \mathcal{E}|_{\mathscr{Y}_0} \to F \to \mathcal{O}_{\mathscr{Y}_0}$ corresponds to a section $s_0^\vee$ of $\mathcal{E}^\vee|_{\mathscr{Y}_0}$ and the derived zero-locus of this section $\mathscr{Z}_0 := \{s_0^\vee = 0\} \hookrightarrow \mathscr{Y}_0$ defines a quasi-smooth closed immersion. Here the derived zero-locus is the pull-back fitting

in the cartesian square

$$
\begin{array}{ccc}
\mathscr{Z}_0 & \longrightarrow & \mathscr{Y}_0 \\
\downarrow & \square & \downarrow s_0^\vee \\
\mathscr{Y}_0 & \xrightarrow{\ 0\ } & \mathbb{V}(\mathcal{E}|_{\mathscr{Y}_0}).
\end{array}
$$

The map $s_0 \colon \mathcal{E}|_{\mathscr{Y}_0} \to F$ corresponds to a 2-commutative diagram

$$
\begin{array}{ccc}
\mathscr{W}_0 & \longrightarrow & \mathscr{Y}_0 \\
\downarrow & \nearrow & \downarrow s_0^\vee \\
\mathscr{Y}_0 & \xrightarrow{\ 0\ } & \mathbb{V}(\mathcal{E}|_{\mathscr{Y}_0}).
\end{array}
$$

and hence to a map $\mathscr{W}_0 \to \mathscr{Z}_0$. By construction, we have that $\mathbb{L}_{\mathscr{W}_0/\mathscr{Z}_0} = 0$ so the closed immersion $\mathscr{W}_0 \to \mathscr{Z}_0$ is also an open immersion. After replacing $\mathscr{Y}$ with an open neighborhood of $\mathscr{W}_0$, we can thus assume that $\mathscr{W}_0 = \mathscr{Z}_0$.

Finally, we may lift the section $s_0^\vee$ of $\mathcal{E}^\vee|_{\mathscr{Y}_0}$ to a section $s^\vee$ of $\mathcal{E}^\vee$ since $\mathscr{Y}$ is cohomologically affine. The derived zero-locus $\mathscr{W} := \{s^\vee = 0\} \hookrightarrow \mathscr{Y}$ is a quasi-smooth closed immersion restricting to $\mathscr{W}_0 \hookrightarrow \mathscr{Y}_0$ and the composition $f \colon \mathscr{W} \hookrightarrow \mathscr{Y} \to \mathscr{X}$ is a quasi-smooth morphism such that $f|_{\mathscr{X}_0} \simeq f_0$. $\square$

APPENDIX A. NON-EXISTENCE OF ZARISKIFICATION

In this section, we show that the Zariskification, in contrast to the henselization, does not exist in general. This counter-example was mentioned in [TT17, 3.1.2].

Let X be a scheme and $Z \hookrightarrow X$ be a closed subscheme. The generization of Z is the subset of X consisting of all points $x \in |X|$ such that $\overline{\{x\}} \cap Z \neq \emptyset$. A *Zariskification* of X along Z is a flat quasi-compact monomorphism $W \to X$ such that the image is the generization of Z. The Zariskification is unique up to isomorphism since if W and W' are two monomorphisms as above, then $W \times_X W' \to W$ and $W \times_X W' \to W'$ are faithfully flat quasi-compact monomorphisms, hence isomorphisms.

If $X = \operatorname{Spec} A$ is an affine scheme and $Z = \operatorname{Spec}(A/I)$ is a closed subscheme, then the Zariskification exists and equals $W = \operatorname{Spec}\big((1+I)^{-1}A\big)$ [Ray70, §2]. If $Z = \{x_1, x_2, \ldots, x_n\}$ is a finite set of points, then the Zariskification is the semi-localization at Z.

In the following example we show that the Zariskification at two points of Hironaka's non-projective proper smooth threefold does not exist.

Example A.1 (Non-existence of Zariskification)**.** Let X be a projective threefold and c, d curves as in [Har77, p. 443]. Let X' be the non-projective proper threefold given by gluing the different blow-ups and let l_0, m_0, l_0' and m_0' be curves on X' as in *loc. cit.* and let $P' = l_0 \cap m_0$ and $Q' = l_0' \cap m_0'$.

There is no affine neighborhood containing both P' and Q'. We claim that the generization E of P' and Q' is not pro-open (i.e., not represented by a flat quasi-compact monomorphism). For this, we can use Raynaud's criterion for locally factorial schemes [Ray68, Cor. 2.7]. Hence, it is enough to show that there is a point x' not in E such that every divisor containing x' intersects E (i.e., intersects P' or Q'). But since $l_0 + m_0'$ is numerically trivial, every divisor that intersects l_0 properly contains m_0'. In particular, every divisor intersecting l_0 contains either P' or Q'. Raynaud's criterion is thus not satisfied for a point x' on l_0 (not equal to P').

Example A.2 (Algebraic space without Zariski-localization at a point)**.** For a suitable choice of X and curves c, d, one can endow Hironaka's proper threefold X'

with a free action of $G = \mathbb{Z}/2\mathbb{Z}$ that interchanges P' and Q'. The quotient X'/G is then not a scheme since the image of $\{P', Q'\}$ is a point z that does not admit an affine neighborhood. Moreover, the Zariskification at z does not exist. Indeed, if there is a flat monomorphism $W \to X'/G$ of algebraic spaces with image the generization of z, then it pulls-back to a flat monomorphism $W' \to X'$ with image the generization of P' and Q'. Since W' is a scheme (see [SP, 0B8A] or [TT17, Thm. 3.1.5]), this contradicts Example A.1.

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