On Averaging Technique for Nonlinear Time-Varying Systems without Local Lipschitz Continuity

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Abstract: The majority of literature for averaging methods deal with a class of nonlinear time-varying (NLTV) dynamics and their time-invariant averaged systems with locally Lipschitz continuous (LLC) condition. In this work, a class of uniformly continuous NLTV systems and their uniformly continuous averaged systems are considered. With a careful discussion of the existence of solutions, the first result shows the closeness of solutions between the original NLTV system and its averaged system with a sufficiently small time-scale separation parameter ε on a subset of a time interval, in which both the NLTV system and the averaged system have well-defined solutions. If the averaged system is finite-time stable, the second result shows that the original NLTV system will uniformly converge to an arbitrarily small neighborhood of the origin on a finite-time interval with sufficiently small ε. Simulation results support the theoretical finding.

Keywords: Nonlinear systems, Averaging, Finite-time stability, Closeness of solutions, Non-Lipschitz continuity

1. INTRODUCTION

Averaging theory is one of the most widely used tools to analyse the stability property of a class of parameterized nonlinear time-varying (NLTV) systems from the stability property of their time-invariant averaged systems. It has been used in various applications, such as extremum seeking control Krstić and Wang (2000); Tan et al. (2006), parameter identification and adaptive control Anderson et al. (1986); Solo and Kong (1994), vibrational control Bellman et al. (1986); Cheng et al. (2018), power electronic system Krein et al. (1990); Lehman and Bass (1996) and so on.

Most averaging methods deal with a class of NLTV dynamics and their time-invariant averaged systems when the nonlinear mappings are locally Lipschitz continuous (LLC) Khalil (2002); Nesic and Teel (2001); Sanders and Verhulst (2007); Teel et al. (1999). On one hand, locally Lipschitz continuity provides the uniqueness of solutions of nonlinear dynamics on an interval close to the initial time instant. On the other hand, LLC condition guarantees a limited rate of change for the nonlinear mapping on any compact set. Such properties are necessary for the classical averaging analysis including the traditional method originated from the work of Bogoliubov N.N. Bogoliubov (1961), Khalil (2002), the trajectory-based method Aeyels and Peuteman (1998); Sanders and Verhulst (2007), and Lyapunov method Teel et al. (1999); Wang and Nesic (2010).

However, non-Lipschitz nonlinear dynamics appear quite often in engineering applications. For example, the sliding mode control is a well-known nonlinear control method that alters the dynamics of a nonlinear system by applying non-smooth control algorithms Basin (2019); Boiko and Fridman (2005) to achieve finite reaching time to the sliding surface. Recently, the concept of finite-time stability has gained a lot of attention. As indicated in Bhat and Bernstein (2000), finite-time stability is usually achieved for a class of continuous but non-Lipschitz systems. Compared with asymptotic stability, achieving the desired performance in a finite-time interval is more attractive in engineering applications, see, for example, finite-time stabilization for controllable systems Hong (2002), finite-time optimization Garg and Panagou (2021) with applications Poveda and Krstić (2021); Ríos et al. (2017).

Averaging techniques that assume a finite-time stable averaged system without LLC condition have not been well studied. An averaging tool to deal with a class of non-Lipschitz NLTV systems was developed by Russian mathematics in Kranosel'skii and Krein (1955). However, only non-uniform closeness of solutions results over a finite-time interval was discussed without any stability analysis for the NLTV systems. A few papers including Arstein (1998a); Wang et al. (2012) studied a class of time-invariant singularly perturbed systems without LLC condition via averaging along the trajectories of the fast dynamics to conclude some asymptotic behaviors, though this paper focuses on investigating some finite-time stability properties for a class of NLTV system via its finite-time stable averaged system. In Poveda and Krstić (2021), the fixed-time extremum seeking algorithms were proposed by using Wang et al. (2012) directly without rigorous analysis.

This work considers a class of NLTV systems following the form in (Khalil, 2002, Chapter 10) without the LLC condition. With the maximum interval of existence of solutions for the NLTV system and its averaged system carefully addressed, the first result shows the closeness of solutions for two systems uniformly in the initial time t_0 over a subset of the common maximum interval of existence of solutions. Based on the first result, the second result shows that given the origin of the averaged system is finite-time stable, the original NLTV system will uniformly converge to an arbitrarily small neighborhood of the origin over a finite-time interval by tuning the time-scale separation parameter ε sufficiently small.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Preliminaries

The notation $\mathcal{R}$ represents the set of all real numbers and $\mathcal{N}$ represents the set of all integers. For any vector $\mathbf{x} \in \mathcal{R}^n$, $|\mathbf{x}|$ represents its Euclidean norm, which is defined as $|\mathbf{x}| \triangleq \sqrt{\mathbf{x}^\intercal \mathbf{x}}$, where $(\cdot)^\intercal$ represents the transpose.

A continuous function $\alpha : \mathcal{R}_{\geq 0} \to \mathcal{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is zero at zero and strictly increasing. A continuous function $\sigma : \mathcal{R}_{\geq 0} \to \mathcal{R}_{\geq 0}$ is said to be of class $\mathcal{L}$ if it is converging to zero as its argument grows unbounded. A continuous function $\tilde{\alpha} : \mathcal{R}_{\geq 0} \to \mathcal{R}_{\geq 0}$ is said to be a generalized $\mathcal{K}$ function ($G\mathcal{K}$ function) if it satisfies

$$\begin{cases} \tilde{\alpha}(s_1) > \tilde{\alpha}(s_2) & if\ \tilde{\alpha}(s_1) > 0,\ s_1 > s_2, \\ \tilde{\alpha}(s_1) = \tilde{\alpha}(s_2) & if\ \tilde{\alpha}(s_1) = 0,\ s_1 > s_2. \end{cases} \quad (1)$$

A continuous function $\beta : \mathcal{R}_{\geq 0} \times \mathcal{R}_{\geq 0} \to \mathcal{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. A continuous function $\tilde{\beta} : \mathcal{R}_{\geq 0} \times \mathcal{R}_{\geq 0} \to \mathcal{R}_{\geq 0}$ is a generalized $\mathcal{KL}$ function ($G\mathcal{KL}$ function) if, for each fixed $t \geq 0$, the function $s \to \tilde{\beta}(s, t)$ is a generalized $\mathcal{K}$ function, and for each fixed $s \geq 0$, the function $t \to \tilde{\beta}(s, t)$ decreases to zero as $t \to T$ for some $T < \infty$.

Similar to Haddad et al. (2008), a class of nonlinear time-varying (NLTV) systems is considered:

$$\dot{x} = f_0(t, x),\ x(t_0) = x_0 \in D_0,\ t \in [t_0, t_0 + I_{x_0,t_0}), \quad (2)$$

where the nonlinear mapping $f_0(t, x) : \mathcal{R}_{\geq 0} \times D \to \mathcal{R}^n$ is **continuous** with respect to (t, x) with $0 \in D_0 \subseteq D \subseteq \mathcal{R}^n$. $t_0 \geq 0$, and $0 < I_{x_0,t_0} \leq \infty$ is the maximal interval of existence of solutions of (2). Denote the $S(t, t_0, x_0)$ as the set of all the solutions of (2) with the initial state x_0 and initial time t_0.

Remark 1. It is noted that the continuity of f_0 guarantees the existence of solutions, but the solutions might not be unique due to possible lack of local Lipschitz continuity. In Haddad et al. (2008), it was assumed that the system (2) possesses a unique solution in forward time for all initial conditions except possibly the origin. Also, $f_0(t, 0) \equiv 0$ holds for any $t \geq 0$, indicating that the origin is the equilibrium of interests. In this work, the existence of multiple solutions with initial conditions besides the origin for (2) is considered, and the origin of (2) is not necessarily an equilibrium. In particular, our first result (Theorem 1) shows the closeness of solutions between the original NLTV system and its time-invariant averaged system over a finite-time interval without any stability requirements or forward uniqueness of solutions. Therefore, the requirement of the system (2) is weaker than that in Haddad et al. (2008). ◦

A simpler subclass of time-varying systems (2) is a time-invariant system:

$$\dot{x} = f_0(t, x) = f_1(x),\ x(0) = x_0 \in D_0,\ t \in [0, I_{x_0}), \quad (3)$$

with the set of all the solutions of (3) denoted as $S(t, x_0)$. I_{x_0} is the maximal interval of existence of solutions of (3).

2.2 Problem formulation

This paper focuses on a family of NLTV systems parameterized by a small positive parameter ε:

$$\dot{x} = f\left(\frac{t}{\varepsilon}, x\right),\ x(t_0) = x_0 \in D_0,\ t \in [t_0, t_0 + I^a_{x_0,t_0,\varepsilon}], \quad (4)$$

with $\varepsilon \in (0, \varepsilon_0)$ for some positive ε_0 and $t_0 \geq 0$. The state satisfies $x \in D \subseteq \mathcal{R}^n$ and D_0 is a compact subset of the compact set D, where $0 \in D_0$. The set $S_\varepsilon(t, t_0, x_0)$ contains all solutions of (4) with any solution denoted as $x_\varepsilon(t, t_0, x_0)$.

The following assumptions are used in this work. The first assumption is about the property of the nonlinear mapping $f(\cdot, \cdot)$.

Assumption 1. The nonlinear mapping $f(\tau, x) : \mathcal{R}_{\geq 0} \times D \to \mathcal{R}^n$ is uniformly continuous with respect to x, uniformly in τ. Moreover, for any $x \in D$, $f(\tau, x)$ is continuous and uniformly bounded in τ. □

From Assumption 1, for simplicity of presentation, we assume that $f(\tau, x)$ is uniformly continuous with respect to x compared with the continuity assumption for (2). This assumption is sufficient in most cases when dealing with nonlinear dynamics with finite-time stability (see Garg and Panagou (2021); Hong (2002)). Also, for switched systems Wang and Nesic (2010), $f(\tau, x)$ is piecewise continuous in τ while we assume $f(\tau, x)$ is continuous in τ in Assumption 1 for simplicity. We will further relax these in our future work.

Remark 2. Most existing averaging techniques require that the nonlinear mapping $f(\tau, x)$ is locally Lipschitz continuous (LLC) with respect to x, uniformly in τ Khalil (2002); Sanders and Verhulst (2007); Teel and Nesic (2000). Also, it was assumed that the solutions are forward complete [1] in Teel and Nesic (2000). Without the LLC condition, the NLTV system (4) might have multiple solutions and different convergence property compared with LLC systems. Also, the existence of solutions on a certain time interval needs to be carefully addressed. ◦

The key idea of averaging techniques is to conclude some uniform stability properties with respect to time "t" for the NLTV dynamics (4) from its time-invariant averaged

[1] The definition of forward completeness can be found in (Teel and Nesic, 2000, Definition 3).

system with a sufficiently small parameter ε. To achieve this, the closeness of solutions of two systems over a finite time interval needs to be analysed first. Two assumptions are thus needed. One is the existence of a well-defined averaged system. The other is the existence of a common interval of the existence of solutions for two systems.

Assumption 2. There exists a uniformly continuous function $f_{av}(x) : D \to \mathcal{R}^n$ such that there exist $\beta_{av} \in \mathcal{KL}$ and $T^* > 0$, the following inequality holds

$$\left| f_{av}(x) - \frac{1}{T}\int_{t_0}^{t_0+T} f(\tau, x)d\tau \right| \leq \beta_{av}(\max\{|x|, 1\}, T), \quad (5)$$

for any $T > T^*$, $t_0 \geq 0$, $x \in D$. □

It is noted that $f_{av}(\cdot)$ is also uniformly continuous and not necessarily LLC. Assumption 2 defines the following averaged system of (4):

$$\dot{x} = f_{av}(x),\ x(t_0) = x_0 \in D_0,\ t \in [t_0, t_0 + I^b_{x_0}]. \quad (6)$$

The notion of $S_{av}(t, x_0)$ denotes the set of all solutions of (6), and $x_{av}(t - t_0, x_0)$ is any solution in $S_{av}(t, x_0)$ which is not dependent on t_0.

Assumption 3. For any $x_0 \in D_0$, $t_0 \geq 0$, $\varepsilon \in (0, \varepsilon_0)$, there exists $0 < I^* \leq \min\{I^a_{x_0,t_0,\varepsilon}, I^b_{x_0}\}$ such that all solutions of (4), (6) satisfy $S_\varepsilon(t, t_0, x_0) \subseteq D$ and $S_{av}(t, x_0) \subseteq D$ for any $t \in [t_0, t_0 + I^*]$. □

Remark 3. Assumption 3 shows the boundedness for all solutions of two systems (4), (6) (see a similar assumption in (Deghat et al., 2021, Assumption 1)). In particular, such boundedness property holds for all solutions of the original system (4) uniformly in x_0, t_0, ε. This assumption plays an important role in showing the closeness of solutions on a finite interval in the next section. How to relax this assumption will be addressed in our future work. ○

3. CLOSENESS OF SOLUTIONS ON A FINITE TIME INTERVAL

This section discusses the closeness of solutions between the original system (4) and its averaged system (6) on a finite-time interval $[t_0, t_0 + I]$ for any $0 < I \leq I^*$, $t_0 \geq 0$. Now each solution of systems (4) and (6) is a continuous function $x_\varepsilon(t, t_0, x_0) : [t_0, t_0 + I] \to D$ and $x_{av}(t - t_0, x_0) : [t_0, t_0 + I] \to D$ respectively. It is noted that the result in this section do not require any stability property or uniqueness of solutions for the original system (4) or its averaged system (6). Furthermore, such closeness of solutions analysis plays a key role in the finite-time stability analysis in the next section.

It is highlighted that the widely used proof technique for averaging methods Nesic and Teel (2001); Sanders and Verhulst (2007) is based on the Gronwall Lemma (Khalil, 2002, Lemma A.1), which requires LLC condition. Since the nonlinear mappings $f(\cdot, \cdot)$, $f_{av}(\cdot)$ are not necessarily LLC, new proof techniques are needed. In this work, the proof is completed based on Arzela-Ascoli theorem (Bressan, 2013, Corollary 3.13). Motivated by (Arstein, 1998b, Lemma 4.3), the following two lemmas are used to prove the first main result.

Lemma 1. Suppose Assumptions 1 and 3 hold. Given any $I \in (0, I^*]$, for each solution $x_\varepsilon(t, t_0, x_0)$ of the original system (4), there exists a uniformly continuous function $y(t, t_0, x_0) : [t_0, t_0 + I] \to D$ with $y(t_0, t_0, x_0) = x_0$, such that for any $\delta > 0$, there exists a $\varepsilon_1^* > 0$ such that for any $\varepsilon \in (0, \varepsilon_1^*)$, the following inequality holds

$$|x_\varepsilon(t, t_0, x_0) - y(t, t_0, x_0)| < \delta, \quad (7)$$

for any $x_0 \in D_0$, $t \in [t_0, t_0 + I]$, $t_0 \geq 0$.

The proof of Lemma 1 is provided in Appendix A. Lemma 1 shows that with $\varepsilon \to 0$, each solution $x_\varepsilon(t, t_0, x_0)$ of the original system (4) converges to a function $y(t, t_0, x_0)$, which is independent of ε, uniformly in (t_0, x_0).

Lemma 2. Suppose Assumptions 1-3 hold. Let $x_\varepsilon(t, t_0, x_0)$ and $y(t, t_0, x_0)$ come from Lemma 1. For any $\delta > 0$, there exists a $\varepsilon_2^* > 0$ such that for any $\varepsilon \in (0, \varepsilon_2^*)$, the following inequality holds

$$\left| \int_{t_0}^{t} f\left(\frac{s}{\varepsilon}, x_\varepsilon(s, t_0, x_0)\right) ds - \int_{t_0}^{t} f_{av}(y(s, t_0, x_0))ds \right| < \delta, \quad (8)$$

for any $x_0 \in D_0$, $t \in [t_0, t_0 + I]$, $t_0 \geq 0$.

The proof of Lemma 2 is provided in Appendix B. By combining two convergence properties in Lemma 1 and Lemma 2, the function $y(t, t_0, x_0)$ is shown to be a solution $x_{av}(t-t_0, x_0)$ of the averaged system (6). This leads to the following theorem which presents the closeness of solutions of two systems (4) and (6) on a given finite-time interval $[t_0, t_0 + I]$.

Theorem 1. Suppose Assumptions 1-3 hold. Given any $I \in (0, I^*]$, for any $\delta > 0$, there exists a small $\varepsilon^* > 0$ such that for any $0 < \varepsilon < \varepsilon^*$, for each solution $x_\varepsilon(t, t_0, x_0)$ of the original system (4), there exists a solution $x_{av}(t - t_0, x_0)$ of the averaged system (6) such that

$$|x_\varepsilon(t, t_0, x_0) - x_{av}(t - t_0, x_0)| < \delta, \quad (9)$$

holds for any $x_0 \in D_0$, $t \in [t_0, t_0 + I]$, and $t_0 \geq 0$.

Proof: From Assumption 1, $f(\tau, x)$ is continuous with respect to τ and x, and uniformly bounded with respect to τ. From Assumption 3, $x_\varepsilon(t, t_0, x_0) \in D$ holds for $t \in [t_0, t_0 + I]$ uniformly in x_0, t_0, ε. Therefore, each solution $x_\varepsilon(t, t_0, x_0)$ of the system (4) is continuous differentiable for any $x_0 \in D_0$, $t_0 \geq 0$, $\varepsilon \in (0, \varepsilon_0)$.

Since $x_\varepsilon(t, t_0, x_0)$ is a solution of the system (4), for any $\varepsilon \in (0, \varepsilon_0)$, it has

$$x_\varepsilon(t, t_0, x_0) - x_0 = \int_{t_0}^{t} f\left(\frac{s}{\varepsilon}, x_\varepsilon(s, t_0, x_0)\right) ds, \quad (10)$$

for any $x_0 \in D_0$, $t \in [t_0, t_0 + I]$, $t_0 \geq 0$. From the convergence results in inequalities (7) and (8), by using equation (10), with $\varepsilon \to 0$, it follows that

$$y(t, t_0, x_0) - x_0 = \int_{t_0}^{t} f_{av}(y(s, t_0, x_0))ds, \quad (11)$$

for any $x_0 \in D_0$, almost all $t \in [t_0, t_0 + I]$, any $t_0 \geq 0$. The equation (11) implies that $y(t, t_0, x_0)$ is a solution of the averaged system (6), namely $y(t, t_0, x_0) := x_{av}(t - t_0, x_0)$, which is not related with t_0 due to the time-invariant property of (6). By applying Lemma 1, the results of Theorem 1 can be obtained with $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*\}$. This completes the proof. □

Remark 4. Theorem 1 indicates that by choosing a sufficiently small positive ε, for each solution of the original system (4), there exists a solution of the averaged system (6) such that two solutions are arbitrarily close for a given time interval $[t_0, t_0 + I]$, for any $0 < I \leq I^*$, $t_0 \geq 0$. This result is similar to that in classic averaging techniques with LLC condition in (Sanders and Verhulst, 2007, Theorem 2.81), (Teel and Nesic, 2000, Theorem 1). By comparison, for our cases, two systems: (4) and (6) might possess multiple solutions. Moreover, the interval I cannot be arbitrarily selected as in the existing results with forward completeness property. ◦

4. FINITE-TIME STABILITY

This section applies the result obtained in Theorem 1 to conclude the finite-time practical stability of the original system (4) from the finite-time stability of its averaged system (6). Firstly, the finite-time stability for the time-invariant system (3) is defined.

Definition 1. (Bhat and Bernstein, 2000, Definition 2.2) The origin of the system (3) is said to be finite-time stable if it is Lyapunov stable and finite-time convergent. Moreover, there exists a continuous settling-time function $\hat{T}(x_0)$ such that

$$\lim_{t \to \hat{T}(x_0)} x(t, x_0) = 0, \tag{12}$$

where $x(t, x_0)$ is the solution of (3).

In Haddad et al. (2008), the finite-time stability of the origin for NLTV system (2) is also defined with a settling-time function $\hat{T}(t_0, x_0)$. In this work, the "settling time" to reach a small neighborhood of the origin for the original system (4) is uniform in t_0 if its averaged system (6) is finite-time stable (see Theorem 2). Thus the definition of finite-time stability for (3) system is not provided here.

Next proposition shows the equivalence between the definition of finite-time stability and its characterization by using a class $\mathcal{GKL}$ function.

Proposition 1. The origin of the system (3) is finite-time stable if and only if for any initial state $x_0 \in D_0$, the solution $x(t, x_0)$ of (3) satisfies:

$$|x(t, x_0)| \leq \tilde{\beta}(|x_0|, t), \tag{13}$$

for any $t \geq 0$. $\tilde{\beta}(\cdot, \cdot)$ is a $\mathcal{GKL}$ function with $\tilde{\beta}(x_0, t) \equiv 0$ when $t \geq \hat{T}(x_0)$ with $\hat{T}(x_0)$ continuous with respect to x_0 and $\hat{T}(0) = 0$. $\hat{T}(x_0)$ is the settling-time function.

The proof of Proposition 1 follows the similar proof for the uniformly asymptotic stability property in (Khalil, 2002, Lemma 4.5). The proof is omitted due to space limitations.

Next result shows how to conclude the stability property of the original system (4) provided that the averaged system (6) is finite-time stable.

Theorem 2. Suppose Assumptions 1-3 hold, and the origin of the averaged system (6) is finite-time stable with a domain of attraction containing D_0 with the settling-time function $\hat{T}(\cdot)$ and $\tilde{\beta}(\cdot, \cdot) \in \mathcal{GKL}$ in Proposition 1. Then the origin of the original system (4) is uniformly practically finite-time stable in ε. That is, for any $\delta > 0$, there exists a small $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$, each solution $x_\varepsilon(t, t_0, x_0)$ of the original system (4) satisfies

$$|x_\varepsilon(t, t_0, x_0)| \leq \tilde{\beta}(|x_0|, t - t_0) + \delta, \tag{14}$$

for any $x_0 \in D_0$, $t \geq t_0 \geq 0$.

Sketch of Proof:

From the finite-time stability of (6), $\tilde{\beta}(|x_0|, \hat{T}(x_0)) = 0$, $x_{av}(t - t_0, x_0) = 0$ at $t = t_0 + \hat{T}(x_0)$, and $I^b_{x_0} = \infty$. We consider two arbitrarily small positive constant δ_1, δ_2 that satisfies $\tilde{\beta}(\delta_2, 0) \leq \delta_1$. From the closeness of solutions result in Theorem 1, considering the finite-time interval $[t_0, t_0 + \hat{T}(x_0)]$, with sufficiently small ε_1^*, for any $\varepsilon \in (0, \varepsilon_1^*)$, $|x_\varepsilon(t, t_0, x_0) - x_{av}(t - t_0, x_0)| \leq \delta_2$. This leads to $|x_\varepsilon(t, t_0, x_0)| \leq \delta_2$ at $t = t_0 + \hat{T}(x_0)$.

We initialize $x_{av}(t - t_0, x_0) = x_\varepsilon(t, t_0, x_0)$ at $t = t_0 + \hat{T}(x_0)$, With sufficiently small ε_2^*, the closeness of solutions holds $|x_\varepsilon(t, t_0, x_0) - x_{av}(t - t_0, x_0)| \leq \delta_2$ for $t \in [t_0 + \hat{T}(x_0), t_0 + 2\hat{T}(x_0)]$. Since $|x_{av}(t - t_0, x_0)| \leq \tilde{\beta}(\delta_2, t - t_0) \leq \delta_1$, we have $|x_\varepsilon(t, t_0, x_0)| \leq \delta_1 + \delta_2$ for $t \in [t_0 + \hat{T}(x_0), t_0 + 2\hat{T}(x_0)]$, and $|x_\varepsilon(t, t_0, x_0)| \leq \delta_2$ at $t = t_0 + 2\hat{T}(x_0)$.

By induction, we have $|x_\varepsilon(t, t_0, x_0)| \leq \delta_1 + \delta_2$ for $t \geq t_0 + \hat{T}(x_0)$. Therefore, with $\varepsilon^* = \min(\varepsilon_1^*, \varepsilon_2^*)$ and $\delta = \delta_1 + \delta_2$, $|x_\varepsilon(t, t_0, x_0)| \leq \tilde{\beta}(|x_0|, t - t_0) + \delta_1$ for $t \in [t_0, t_0 + \hat{T}(x_0)]$, and $|x_\varepsilon(t, t_0, x_0)| \leq \delta$ for $t \geq t_0 + \hat{T}(x_0)$. Thus inequality (14) holds, completing the proof. □

Remark 5. The uniformly practical finite-time stability in Theorem 2 is an extension of the uniformly practical asymptotic stability in (Nesic and Teel, 2001, Definition 6). For the uniformly practical finite-time stability, each solution of the original system (4) converges to an arbitrarily small neighborhood of the origin at the settling time $\hat{T}(x_0)$ of its averaged system (6), namely $\lim_{t \to t_0 + \hat{T}(x_0)} |x_\varepsilon(t, t_0, x_0)| \leq \delta$. It is noted that the "settling time" is independent of t_0 and small ε. ◦

5. SIMULATION

This section presents two numerical examples to illustrate two results. The first example shows the closeness of solutions of the original system (4) and the averaged system (6) on a finite-time interval without any stability assumption (Theorem 1). The second example demonstrates the stability of the original system (4) when the averaged system (6) is finite-time stable (Theorem 2).

5.1 Example 1

Consider the following system

$$\dot{x} = x^{\frac{1}{3}} + \sin\left(\frac{t}{\varepsilon}\right), \tag{15}$$

where ε is a small positive parameter. Assumption 1 holds for (15). It is also checked that the following averaged system

$$\dot{x}_{av} = x_{av}^{\frac{1}{3}}, \tag{16}$$

satisfies Assumption 2. The initial state of (15) and (16) is chosen as $x_0 = 0$. It is noted that the origin of (16) is a finite-time repeller (see (Bhat and Bernstein, 2000, Definition 3.1)) with two solutions $x_{av}(t, 0) = (2t/3)^{3/2}$ and $x_{av}(t, 0) \equiv 0$. This shows that (15) and (16) are forward complete, but unstable. Therefore, Assumption 3 holds, and we choose the finite-time interval as $I = 5$.

Fig.1 shows the solution $x_{av}(t, 0) = (2t/3)^{3/2}$ of the averaged system (16), and the corresponding solutions of the original system (15) with $\varepsilon = 0.05$ and 0.2 respectively. It shows that when a smaller ε is used, the trajectory of the original system is closer to the trajectory of the averaged system. This indicates that the simulation results are consistent with Theorem 1.

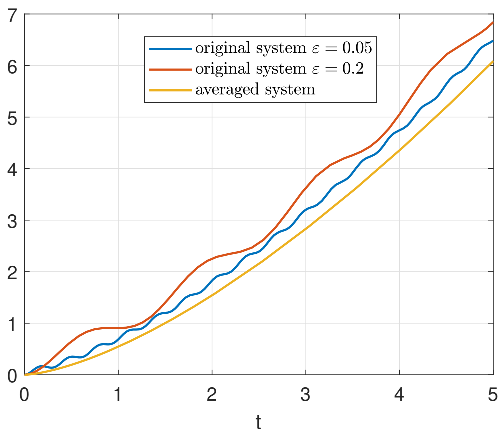

Fig. 1. Solutions of the original system (15) with $\varepsilon = 0.05$ and 0.2, and the corresponding solution of the averaged system (16)

5.2 Example 2

Now the following system is considered

$$\dot{x} = -\sqrt{|x|}\text{sign}(x) + \sin\left(\frac{t}{\varepsilon}\right)(1 + x). \tag{17}$$

where sign$(\cdot)$ is the sign function. The averaged system of (17) can be written as

$$\dot{x}_{av} = -\sqrt{|x_{av}|}\text{sign}(x). \tag{18}$$

Similar to Example 1, it is checked that Assumption 1-3 hold. The stability analysis of the averaged system (18) can be found in (Bhat and Bernstein, 2000, Example 2.1), indicating the origin of (18) is finite-time stable. The initial state of systems (17) and (18) is selected to be $x_0 = 2$. The settling time of the system (18) can be calculated: $\hat{T}(x_0) = 2|x_0|^{0.5} = 2.83$ (see more details in Bhat and Bernstein (2000)). We select $\varepsilon = 0.05$ and 0.15 respectively. As shown in Fig.2, the trajectories of the original system with different ε can both finite-time converge to a small neighborhood of the origin. The size of the neighborhood is getting smaller when a smaller ε is used. The simulation results are consistent with the results in Theorem 2.

6. CONCLUSION

This paper investigated the closeness of solutions between a class of nonlinear time-varying (NLTV) systems and their time-invariant averaged systems when neither of them satisfies LLC condition. This closeness of solutions result can be used to conclude that the trajectories of the

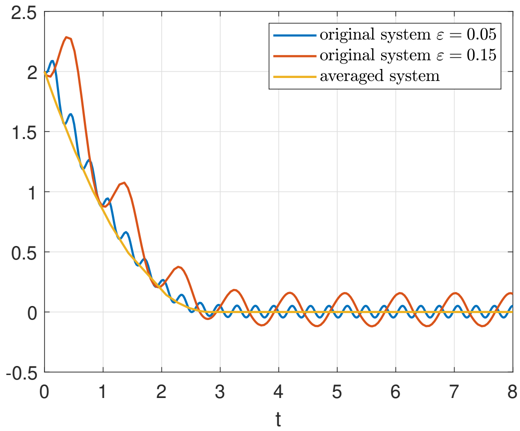

Fig. 2. Solutions of the original system (17) with $\varepsilon = 0.05$ and 0.15 respectively and the solution of the averaged system (18)

NLTV system converge to a small neighbourhood of the origin on a finite-time interval by tuning the time-scale parameter ε sufficiently small when its averaged system is finite-time stable. Our future work will further extend these results to a more general setting such as weaker continuity conditions.

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Appendix A. PROOF OF LEMMA 1

From Assumption 1, $f(\tau, x)$ is bounded for any $\tau \geq 0$, $x \in D$ where D is a compact set. Here we denote a constant $0 < B < \infty$ as the upper bound such that $|f(\tau, x)| \leq B$, for any $\tau \geq 0$, $x \in D$. From Assumption 1 and 3, for any $t_0 \geq 0$, $x_0 \in D_0$, ε-parametrized functions (solutions of (4)) $x_\varepsilon(t, t_0, x_0) : [t_0, t_0 + I] \to D$ are continuously differentiable for $\varepsilon \in (0, \varepsilon_0)$.

For any $x_0 \in D_0$, $t_0 \geq 0$, we denote $\mathcal{F}_{t_0,x_0}$ as a (t_0, x_0)-parametrized continuous function space which consists of all functions $x_\varepsilon(t, t_0, x_0)$ with $\varepsilon \in (0, \varepsilon_0)$. Here we use the Arzela-Ascoli theorem (Bressan, 2013, Corollary 3.13) to prove the convergence property in Lemma 1. To apply the Arzela-Ascoli theorem and show the convergence property uniformly in (t_0, x_0), we need to prove the uniform boundedness and uniform equicontinuity of the function space $\mathcal{F}_{t_0,x_0}$ uniformly in (t_0, x_0). Assumption 3 guarantees the uniform boundedness, since for any $x_0 \in D_0$ and $t_0 \geq 0$, $x_\varepsilon(t, t_0, x_0) \in D$ holds for any $t \in [t_0, t_0 + I]$ uniformly in ε and D is compact. Then we assume $\mathcal{F}_{t_0,x_0}$ is uniformly equicontinuous, namely for any $x_0 \in D_0$, $t_0 \geq 0$, at any $t \in [t_0, t_0 + I)$, for any $\delta > 0$, there exists $h > 0$ such that

$$|x_\varepsilon(t + h, t_0, x_0) - x_\varepsilon(t, t_0, x_0)| < \delta, \tag{A.1}$$

for any $\varepsilon \in (0, \varepsilon_0)$. Due to the continuous differentiability of $x_\varepsilon(t, t_0, x_0)$ and the boundedness of $f(\tau, x)$ uniformly in τ and x, we have

$$\begin{aligned} &|x_\varepsilon(t + h, t_0, x_0) - x_\varepsilon(t, t_0, x_0)| \\ =& \int_t^{t+h} |f(\tau, x_\varepsilon(\tau, t_0, x_0))| \, d\tau \leq Bh < \delta. \end{aligned} \tag{A.2}$$

The existence of h ($0 < h < \frac{\delta}{B}$) is proved. Therefore, for any $x_0 \in D_0$, $t_0 \geq 0$, $\mathcal{F}_{t_0,x_0}$ is uniformly equicontinuous.

By Arzela-Ascoli theorem, as $\varepsilon \to 0$, there exists a subsequence $x_{\varepsilon_{k_n}}(t, t_0, x_0) : [t_0, t_0 + I] \to D$ uniformly converges to a limit $y(t, t_0, x_0) : [t_0, t_0 + I] \to D$ for any $x_0 \in D_0$, $t_0 \geq 0$. By using similar contradiction method in Arstein (1998b), it follows that $x_\varepsilon(t, t_0, x_0)$ converges to $y(t, t_0, x_0)$ as $\varepsilon \to 0$ for any $x_0 \in D_0$, $t_0 \geq 0$. Since $x_\varepsilon(t, t_0, x_0)$ is continuously differentiable, the limit $y(t, t_0, x_0)$ is uniformly continuous. This leads to Lemma 1 by choosing a uniform ε_1^* for any $x_0 \in D_0$, $t_0 \geq 0$.

Appendix B. PROOF OF LEMMA 2

From Assumption 1, $f(\tau, x)$ is uniformly continuous with respect to x, uniformly in τ. From Assumption 2, $f_{av}(\cdot)$ is uniformly continuous. Therefore, for any $\varepsilon \in (0, \varepsilon_0)$, any $\delta > 0$, there exists $h > 0$ such that for any $x, y \in D$, it follows

$$\begin{aligned} |x - y| < h \;\Rightarrow\; \left| f(\frac{t}{\varepsilon}, x) - f(\frac{t}{\varepsilon}, y) \right| < \frac{\delta}{4I}, \\ |x - y| < h \;\Rightarrow\; |f_{av}(x) - f_{av}(y)| < \frac{\delta}{4I}, \end{aligned} \tag{B.1}$$

for any $t \geq 0$.

From now on we choose any small $\delta > 0$, and there exists a corresponding $h > 0$ from (B.1). From the uniform continuity of $y(t, t_0, x_0)$ for any $x_0 \in D_0$, $t_0 \geq 0$ in Lemma 1, we can denote $\bar{y}(t, t_0, x_0)$ as a piecewise constant function with a uniform "sampling period" T_0 such that

$$|y(t, t_0, x_0) - \bar{y}(t, t_0, x_0)| < h, \tag{B.2}$$

for any $x_0 \in D_0$, $t \in [t_0, t_0 + I]$, $t_0 \geq 0$.

From Lemma 1, there exists $\varepsilon_{21}^* > 0$, for any $\varepsilon \in (0, \varepsilon_{21}^*)$ such that

$$|x_\varepsilon(t, t_0, x_0) - y(t, t_0, x_0)| < h. \tag{B.3}$$

for any $x_0 \in D_0$, $t \in [t_0, t_0 + I]$, $t_0 \geq 0$.

For any $x_0 \in D_0$, $t \in [t_0, t_0 + I]$, $t_0 \geq 0$, we have

$$\begin{aligned}
&\left|\int_{t_0}^{t} f(\frac{s}{\varepsilon}, x_\varepsilon(s, t_0, x_0))ds - \int_{t_0}^{t} f_{av}(y(s, t_0, x_0))ds\right| \\
\leq &\underbrace{\left|\int_{t_0}^{t} f(\frac{s}{\varepsilon}, x_\varepsilon(s, t_0, x_0))ds - \int_{t_0}^{t} f(\frac{s}{\varepsilon}, y(s, t_0, x_0))ds\right|}_{\mathbf{1}} \\
+ &\underbrace{\left|\int_{t_0}^{t} f(\frac{s}{\varepsilon}, y(s, t_0, x_0))ds - \int_{t_0}^{t} f(\frac{s}{\varepsilon}, \bar{y}(s, t_0, x_0))ds\right|}_{\mathbf{2}} \\
+ &\underbrace{\left|\int_{t_0}^{t} f(\frac{s}{\varepsilon}, \bar{y}(s, t_0, x_0))ds - \int_{t_0}^{t} f_{av}(\bar{y}(s, t_0, x_0))ds\right|}_{\mathbf{3}} \\
+ &\underbrace{\left|\int_{t_0}^{t} f_{av}(\bar{y}(s, t_0, x_0))ds - \int_{t_0}^{t} f_{av}(y(s, t_0, x_0))ds\right|}_{\mathbf{4}}.
\end{aligned} \tag{B.4}$$

Now we turn to bound the right-hand side of (B.4).

Term 1. By using (B.1), (B.3), for any $\varepsilon \in (0, \varepsilon_{21}^*)$, it follows that

$$\begin{aligned}
&\left|\int_{t_0}^{t} f(\frac{s}{\varepsilon}, x_\varepsilon(s, t_0, x_0))ds - \int_{t_0}^{t} f(\frac{s}{\varepsilon}, y(s, t_0, x_0))ds\right| \\
< &\int_{t_0}^{t} \frac{\delta}{4I} \leq \frac{\delta}{4},
\end{aligned} \tag{B.5}$$

for any $t \in [t_0, t_0 + I]$.

Term 2 and Term 4. From (B.1), (B.2), for any $t \in [t_0, t_0 + I]$, it is easy to obtain

$$\begin{aligned}
&\left|\int_{t_0}^{t} f(\frac{s}{\varepsilon}, y(s, t_0, x_0))ds - \int_{t_0}^{t} f(\frac{s}{\varepsilon}, \bar{y}(s, t_0, x_0))ds\right| \\
< &\int_{t_0}^{t} \frac{\delta}{4I} \leq \frac{\delta}{4}.
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
&\left|\int_{t_0}^{t} f_{av}(\bar{y}(s, t_0, x_0))ds - \int_{t_0}^{t} f_{av}(y(s, t_0, x_0))ds\right| \\
< &\int_{t_0}^{t} \frac{\delta}{4I} \leq \frac{\delta}{4}.
\end{aligned} \tag{B.7}$$

Term 3: Define K is the largest positive integer such that $K < \frac{I}{T_0}$. For $k = 0, \ldots, K$, define $t_k = t_0 + kT_0$. Note that for the piecewise constant function $\bar{y}(t, t_0, x_0)$, for each $t_k \leq t \leq t_{k+1}$, $\bar{y}(t, t_0, x_0)$ is a constant. Then for any $t \in [t_0, t_0 + I]$, we have

$$\begin{aligned}
&\left|\int_{t_0}^{t} f(\frac{s}{\varepsilon}, \bar{y}(s, t_0, x_0))ds - \int_{t_0}^{t} f_{av}(\bar{y}(s, t_0, x_0))ds\right| \\
\leq &\sum_{k=0}^{K} \left|\int_{t_k}^{t_k + T_0} f(\frac{s}{\varepsilon}, \bar{y}(s, t_0, x_0)) - f_{av}(\bar{y}(s, t_0, x_0))ds\right|.
\end{aligned} \tag{B.8}$$

Define a new time scale $s_1 = \frac{s}{\varepsilon}$. For any $k = 0, \ldots, K$, by using Assumption 2, there exists a small $\varepsilon_{22}^* > 0$, such that for any $\varepsilon \in (0, \varepsilon_{22}^*)$, we have

$$\begin{aligned}
&\left|\int_{t_k}^{t_k + T_0} f(\frac{s}{\varepsilon}, \bar{y}(s, t_0, x_0)) - f_{av}(\bar{y}(s, t_0, x_0))ds\right| \\
= &\left|\int_{t_k}^{t_k + T_0} f(\frac{s}{\varepsilon}, \bar{y}(s, t_0, x_0))ds - T_0 f_{av}(\bar{y}(s, t_0, x_0))\right| \\
= &\left|\varepsilon \int_{\frac{t_k}{\varepsilon}}^{\frac{t_k}{\varepsilon} + \frac{T_0}{\varepsilon}} f(s_1, \bar{y}(s, t_0, x_0))ds_1 - T_0 f_{av}(\bar{y}(s, t_0, x_0))\right| \\
= &T_0 \left|\frac{\varepsilon}{T_0} \int_{\frac{t_k}{\varepsilon}}^{\frac{t_k}{\varepsilon} + \frac{T_0}{\varepsilon}} f(s_1, \bar{y}(s, t_0, x_0))ds_1 - f_{av}(\bar{y}(s, t_0, x_0))\right| \\
< &T_0 \frac{\delta}{4(I + T_0)}.
\end{aligned} \tag{B.9}$$

From (B.8), (B.9) and $(K + 1)T_0 < I + T_0$, it results in

$$\begin{aligned}
&\left|\int_{t_0}^{t} f(\frac{s}{\varepsilon}, \bar{y}(s, t_0, x_0))ds - \int_{t_0}^{t} f_{av}(\bar{y}(s, t_0, x_0))ds\right| \\
< &(K + 1)\frac{T_0 \delta}{4(I + T_0)} \leq \frac{\delta}{4}.
\end{aligned} \tag{B.10}$$

From the bounds on **term 1-4** on the right-hand side of (B.4), with $\varepsilon_2^* = \min\{\varepsilon_{21}^*, \varepsilon_{22}^*\}$, for any $\varepsilon \in (0, \varepsilon_2^*)$, it follows that

$$\left|\int_{t_0}^{t} f(\frac{s}{\varepsilon}, x_\varepsilon(s, t_0, x_0))ds - \int_{t_0}^{t} f_{av}(y(s, t_0, x_0))ds\right| < \delta, \tag{B.11}$$

for any $x_0 \in D_0$, $t \in [t_0, t_0 + I]$, $t_0 \geq 0$. And the proof for Lemma 2 is completed.