Iterative Learning Control for Linear Time-varying Systems with Input and Output Constraints

Gijo Sebastian[a], Ying Tan[a], Denny Oetomo[a], Iven Mareels[b]

***Abstract*— Due to hardware constraints and safety requirements, many engineering systems have to satisfy input and output constraints. This paper proposes a new feedback-based iterative learning control (ILC) that can ensure the satisfaction of input and output constraints for linear-time-varying (LTV) systems. The proposed control structure consists of an output feedback loop, a feed-forward ILC and a hard constraint for input. A barrier function is used to assist the design of the output feedback in order to satisfy the output constraints. An appropriate saturation function is used in the design of ILC loop to address the input constraints. By using a suitable composite energy function, the main result of this paper shows that the desired trajectory can be learned using the proposed control structure without violating the input and output constraints. Simulation results are presented to demonstrate the effectiveness of the proposed control structure.**

I. INTRODUCTION

Most of the classical control techniques are model-based, which requires sufficient knowledge of the system of interests. Iterative learning control (ILC), on the other hand, is a data-driven or model-free control design methodology for systems that perform repetitive tasks over a finite time interval. By exploiting the knowledge gained through repetitions, ILC design can relax the requirement of model information, leading to a perfect tracking performance over a finite time interval when iteration tends to infinity. It has found wide applications in chemical batch processes, robotic manufacturing, robotic rehabilitation systems and so on, where the tracking task is repetitive in nature (for more details on its applications, see survey papers [1]–[3]).

Many ILC algorithms have a feed-forward structure, which has the form: $u_{i+1}^{ff}(t) = u_i^{ff}(t) + f(e_k(t), \dots e_{k-M}(t)),\ \forall t \in [0,\ T_f], k \leq i, i = 1, 2, \dots$, and M is an integer. Here the control input at $(i+1)^{\text{st}}$ iteration, u_{i+1}^{ff}, is the sum of previous iteration input and a functional of tracking error at previous iterations $e_k,\ e_{k-1}, \dots, e_{k-M}$ for tasks in finite time interval $[0, T_f]$. The beauty of feed-forward ILC is its simplicity in design and implementation. By using either contraction mapping method [4], [5] or composite energy function (CEF) [6], the feed-forward ILC can be applied to a large class of engineering systems with limited knowledge.

When implementing the proposed ILC algorithms in engineering applications, the physical constraints have to be considered. For example, due to constraints from actuators, the input signals are usually within a certain range. The output signals are measured by sensors, which have a range for the measurements. Moreover, sometimes due to safety reasons, some output signals cannot be over a certain range. Such examples include the temperature or pressure in a chemical reactor, the safety region for rehabilitation robotic systems in order to protect patients, the traffic flow of an urban region, etc. Hence it is important to ensure the satisfaction of these constraints in every iteration.

However, many feed-forward ILC algorithms ignore input constraints and output constraints in the design. Even-though the perfect tracking performance can be achieved in steady-state in the iteration domain, during the transient behavior in iteration domain, either output constraints or input constraints might be violated. In ILC literature, constrained optimization based approach in super vector formulation has been used in [7]–[10] to handle constraints in discrete-time systems. In continuous-time systems, input constraints have been handled in [11], [12] and output constraints in [13]–[15] using barrier functions when the state information is available. To the best of author's knowledge, simultaneous satisfaction of both input and output constraints in the standard feed-forward ILC setting for continuous-time systems has not been addressed.

The technique to handle output constraints is to use barrier function (or barrier certificate), which has been widely used in designing feedback control law to handle output constraints, see, for example [16]–[18] and references therein. Generally speaking, when the output constraints are violated, the barrier function will approach to infinity. Thus, if the control input takes the derivative of the barrier function, when the output is close to the output constraints, the control input will drive the output away from the constraint.

It is noted that when introducing the barrier function and its corresponding control law using its derivative, this control law is in the form of output feedback. Intuitively, by incorporating this output feedback with the feed-forward ILC, it is possible to handle both input and output constraints. However, these two control laws might conflict each other in the transient of iteration. Hence, a careful analysis is thus needed.

This paper handles the output-tracking problem in ILC for continuous-time linear-time-varying (LTV) systems in the presence of input and output constraints. In order to simplify the design of ILC, a standard D-type ILC algorithm is used, which can ensure the perfect tracking performance without any constraint for the dynamic system with a relative degree one. A output feedback control law is incorporated with the feed-forward ILC. This output feedback is designed using

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[a]G. Sebastian, Y. Tan and D. Oetomo are with the Melbourne School of Engineering, The University of Melbourne, Parkville VIC 3010 Australia E-mail: gsebastian@student.unimelb.edu.au, {yingt, doetomo} @unimelb.edu.au, [b]I. Mareels is with IBM Research-Australia, 60 City Road, Southbank VIC 3006, Australia Email: imareels@au1.ibm.com

a barrier function, which has a very general form and can prevent violating the output constraints. A hard constraint on input signal is also added to the system. By using a suitable composite energy function, it is shown that the proposed control structure is able to handle both input and output constraints under appropriate assumptions.

II. PRELIMINARIES AND PROBLEM FORMULATION

Firstly, the notations and needed definitions used in this paper are introduced. Let $\mathcal{R}$ denotes the set of real numbers and $\mathcal{N}$ denotes the set of natural numbers. The set of all continuous functions in $[0,\ T_f]$ that are differentiable up to j^{th} order is denoted by $\mathcal{C}^j[0,\ T_f]$ for any $j \in \mathcal{N}$.

For a vector $\mathbf{x} \in \mathcal{R}^n$, $|\mathbf{x}|^2 \triangleq \mathbf{x}^\intercal \mathbf{x}$. A vector is called positive ($\mathbf{x} > 0$) if each element is positive. For any $\mathbf{x}(t) \in \mathcal{C}[0,\ T_f]$, the supremum norm is defined as $\|\mathbf{x}\|_s \triangleq \max_{t\in[0,T_f]} |\mathbf{x}(t)|_\infty$, where $|\mathbf{x}|_\infty = \max_{j\in[1,\dots,n]} |x^j|$ and x^j denotes j^{th} element of $\mathbf{x}$. The $\mathcal{L}^2$ norm is defined as $\|\mathbf{x}\|_{\mathcal{L}^2} \triangleq \left(\int_0^{T_f} |\mathbf{x}(\tau)|^2 d\tau\right)^{\frac{1}{2}}$.

For a given matrix $A \in \mathcal{R}^{n\times m}$, $|A|$ indicates its induced matrix norm. A square matrix $A = A^\intercal > (\geq) 0$ indicates that this matrix is symmetric and positive definite (positive semi-definite). For a square symmetric matrix A, $\lambda_{min}(A)$ stands for its minimum eigenvalue. The notation I_n denotes the identity matrix of dimension n.

Definition 1: A continuous function $\alpha : [0,\ a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$ [19, Definition 4.2].

Definition 2: Let $u \in R$ and $u^* > 0$. The saturation function is defined as $sat(u, u^*) \triangleq sign\,(u)\, min\{u^*, |u|\}$ for any $u \in \mathcal{R}$ where $u^* > 0$ is a scalar constant. For any $\mathbf{u} \in \mathcal{R}^m$ and a positive vector $\mathbf{u}^*$, the saturation function is defined as $sat(\mathbf{u}, \mathbf{u}^*) = \left[sat(u^1, u^{1^*}), \cdots, sat(u^m, u^{m^*})\right]^\intercal$.

The following lemmas are needed to facilitate the proof of main results in this paper.

Lemma 1: [11, Property-3] For any given $\mathbf{u}_r,\ \mathbf{u}$ and $\mathbf{u}^* \in \mathcal{R}^m$ satisfying $sat(\mathbf{u}_r, \mathbf{u}^*) = \mathbf{u}_r$ then the following inequality holds: $|\mathbf{u}_r - sat(\mathbf{u}, \mathbf{u}^*)|^2 \leq |\mathbf{u}_r - \mathbf{u}|^2$. □

Lemma 2: [11, Property-4] For any $\mathbf{u},\ \mathbf{u}^*$ and $\mathbf{w} \in \mathcal{R}^m$ satisfying $\mathbf{u}^* > 0$, if $\mathbf{v} = sat(\mathbf{u}, \mathbf{u}^*) + \mathbf{w}$, then the following inequality holds: $|sat(\mathbf{v}, \mathbf{u}^*) - \mathbf{v}| \leq |\mathbf{w}|$. □

A. A Motivating Example

Consider a simple "pick and place" one-link robotic manipulator from [20], which is a single arm rotating about a base point, with mass M at the end effector. The robot dynamics is given by a second order LTV model:

$$\begin{aligned}\dot{x}_1 =& x_2\\ \dot{x}_2 =& \frac{-b}{I_{rod} + M(t)L^2} x_2 + \frac{K_t}{I_{rod} + M(t)L^2} u,\\ y =& x_2,\end{aligned} \tag{1}$$

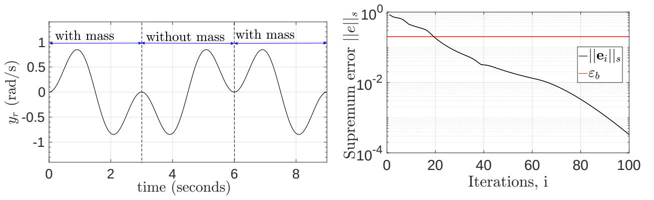
where b is the viscous friction coefficient, K_t is the actuator gain, and L is the length of robot arm. Here, I_{rod} represents the moment of inertia of the arm about the base point. The robot picks and places the mass M during the operation cycle, leading to a LTV system.

Let $a_0(t) = \frac{b}{I_{rod}+M(t)L^2}$ and $a_1(t) = \frac{K_t}{I_{rod}+M(t)L^2}$. When the robot is engaged with mass M, $a_0 = 1$ and $a_1 = 5$ otherwise for the case with $M = 0$, $a_0 = 5$ and $a_1 = 18$.

The reference trajectory is shown in Fig.1. It has been shown that the following ILC law works for the system (1) [6]:

$$u_{i+1}^{ff}(t) = u_i^{ff} + \gamma \dot{e}_i(t),\ \ u_1^{ff}(t) = 0, \tag{2}$$

The simulation is performed for $\gamma = 0.05$ which satisfies the convergence condition given in [6] for systems with a relative degree[1] of one and the re-setting condition (see Assumption 2). The performance requirement needs that the output satisfies $|y(t) - y_d(t)| \leq \varepsilon_b$ with $\varepsilon_b = 0.2$. The supremum norm of tracking error when ILC law (2) is used to evaluate the tracking performance. As shown in Fig. 1, though tracking error converges, the errors in the first 20 iterations violate the output constraint. This shows that a standard ILC is not able to handle output constraints. New ILC laws are needed to take care of input and output constraints simultaneously.



(a) Reference trajectory with operation modes (b) Supremum norm of tracking error

Fig. 1. Reference trajectory and supremum error- motivating example

B. Problem Formulation

Consider a linear time-varying (LTV) multiple-input-multiple-output (MIMO) square system[2] of the following form:

$$\begin{aligned}\dot{\mathbf{x}}(t) =& A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)\\ \mathbf{y}(t) =& C(t)\mathbf{x}(t),\end{aligned} \tag{3}$$

where $\mathbf{x} \in \mathcal{R}^n$ denotes the state and $\mathbf{y},\ \mathbf{u} \in \mathcal{R}^m$ denote the output and input vector respectively. The state matrices $(A(t), B(t), C(t))$ have appropriate dimensions and the elements of the matrices $A(t)$, $B(t)$, $C(t)$ are in $\mathcal{C}[0,\ T_f]$.

The control objective is to design a sequence of control input $\{\mathbf{u}_i(t)\}_{i\in\mathcal{N}}$ such that the output of the system (3) can track a reference output $y_r \in \mathcal{C}^1[0, T]$. Moreover, this system is subjected to hard input and output constraints, which are known a priori. The hard input constraint is the saturation constraint at the total control input (as shown in Fig. 2) whereas the output constraint is defined by $|\mathbf{y}(t)| \leq k_b$, for all $t \in [0, T_f]$ where $k_b > 0$ is the output limit.

The tracking error, $\mathbf{e}(t)$ is defined as $\mathbf{e}(t) = \mathbf{y}_r(t) - \mathbf{y}(t)$.

Remark 1: For a given reference trajectory $\mathbf{y}_r$, the output constraints can be converted to the constraints in terms of

[1] "If the relative degree of system (3) is r, then $CB = CAB = \cdots = CA^{r-2}B = 0_{m\times m}$ and $CA^{r-1}B \neq 0_{m\times m}$" [21, p.387].

[2] A MIMO square system is defined as a system which has the same dimensions for the input and output vectors

tracking error. That is, there exists an $\varepsilon_b > \varepsilon_b^*$ such that if $|\mathbf{e}(t)| \leq \varepsilon_b$, the output constraint: $|\mathbf{y}(t)| \leq k_b$ is satisfied. Such a conversion simplifies the design of feedback back control law at the cost of conservative design of output as pointed out in [15]. $\circ$

The following assumptions are quite standard in ILC.

Assumption 1: There exists a reference input $\mathbf{u}_r \in \mathcal{C}[0,\ T_f]$ and a reference state $\mathbf{x}_r \in \mathcal{C}^1[0,\ T_f]$ for any given reference output $\mathbf{y}_r \in \mathcal{C}^1[0,\ T_f]$ such that the following relationship holds

$$\begin{aligned}\dot{\mathbf{x}}_r(t) =& A(t)\mathbf{x}_r(t) + B(t)\mathbf{u}_r(t)\\ \mathbf{y}_r(t) =& C(t)\mathbf{x}_r(t).\end{aligned} \tag{4}$$

Furthermore, for the given $\mathbf{u}^* > 0$ and $k_b > 0$, it has $sat(\mathbf{u}_r(t), \mathbf{u}^*) = \mathbf{u}_r(t)$ for any $t \in [0,\ T_f]$ and $|\mathbf{y}_r| < k_b$. $\square$

Remark 2: Assumption 1 ensures that there exists a control input $\mathbf{u}_r$ that can track the desired signal $\mathbf{y}_r$. Both reference input and the reference output satisfy the corresponding constraints. $\circ$

Let $(\cdot)_i$ denotes the signal at the i^{th} iteration.

Assumption 2: It is assumed that the system (3) executes a repetitive tracking within a finite time interval $t \in [0,\ T_f]$, and satisfies the identical initial condition: $\mathbf{x}_i(0) = \mathbf{x}_r(0)$ for any iteration $i \in \mathcal{N}$. $\square$

Remark 3: Assumption 2 is a standard assumption in ILC design [6]. This assumption can be relaxed if the perfect tracking performance is not required. $\circ$

Assumption 3: The system (3) has a relative degree of one. In addition, for the simplicity of presentation, it is also assumed that $C(t)B(t) > 0$, for all $t \in [0, T_f]$. $\square$

Remark 4: Assumption 3 is a standard assumption when ILC design is based on Contraction Mapping (CM) based analysis technique. It is possible to accommodate systems with a higher relative degree using appropriate modifications of ILC algorithm. $\circ$

III. CONTROLLER STRUCTURE AND DESIGN

The proposed control structure is shown in Fig 2. It consists of an output feedback loop and a feed-forward loop using an ILC. A hard constraint of input is implemented via an input saturation. The role of the output feedback controller is to ensure that output constraints are satisfied whereas the ILC learns the desired control input in the presence of input constraints. Hence, there are two steps in the design. The first one is to design an output feedback to handle output constraints. The second one is to use a standard ILC control to ensure the perfect tracking with the consideration of input constraints. These would lead to the following control laws:

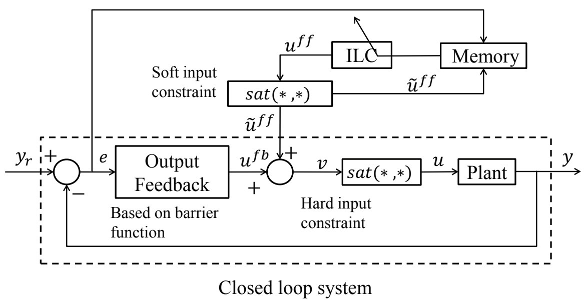


Fig. 2. Block diagram of the proposed control structure

$$\begin{aligned}\mathbf{u}_i(t) =& sat(\mathbf{v}_i(t),\ \mathbf{u}^*)\\ \mathbf{v}_i(t) =& \widetilde{\mathbf{u}}_i^{ff}(t) + \mathbf{u}_i^{fb}(t),\ \forall t \in [0,\ T_f],\end{aligned} \tag{5}$$

where $\widetilde{\mathbf{u}}_i^{ff}(t) = sat(\mathbf{u}_i^{ff}(t),\ \mathbf{u}^*)$ represents the modified input from the ILC control law and $\mathbf{u}_i^{fb}(t)$ is the feedback control.

Remark 5: As the system (3) is LTV, with the bounded control input coming from hard input constraints, the state trajectories are uniformly bounded for any given fixed time interval $t \in [0,\ T_f]$. That is, there exists a compact set $\mathcal{D} \subset \mathcal{R}^n$ such that $\mathbf{x}_i(t) \in \mathcal{D}$, for any $i \in \mathcal{N}$ and any $t \in [0, T_f]$. It is noted that the control objective is to ensure the perfect output tracking with the bounded state trajectories. Hence, the output feedback is sufficient to ensure the perfect tracking. $\circ$

As the system has the relative degree one, for simplicity, the following D-type feed-forward ILC is used:

$$\mathbf{u}_{i+1}^{ff}(t) = \widetilde{\mathbf{u}}_i^{ff}(t) + \Gamma(t)\dot{\mathbf{e}}_i(t),\ \mathbf{u}_1^{ff}(t) = 0, \tag{6}$$

where $\Gamma(t) \in \mathcal{R}^{m\times m} > 0$ is the learning gain. If $\Gamma(t)$ is designed to satisfy some convergence condition, the perfect tracking performance can be achieved when there is no output constraints [5], [6].

Next will provide the design of designing output feedback control law.

A. Design of output feedback in presence of output constraints

Let the state tracking error: $\delta\mathbf{x}(t) = \mathbf{x}_r(t) - \mathbf{x}(t)$. The error dynamics can be obtained from (3) and (4) as

$$\begin{aligned}\delta\dot{\mathbf{x}}(t) =& A(t)\delta\mathbf{x}(t) + B(t)\delta\mathbf{u}(t)\\ \mathbf{e}(t) =& C(t)\delta\mathbf{x}(t)\end{aligned} \tag{7}$$

where $\delta\mathbf{u}(t) = \mathbf{u}_r(t) - \mathbf{u}(t)$ where $\mathbf{x}_r$ and $\mathbf{u}_r$ come from Assumption 1.

The output feedback is designed on the basis of some barrier function. When output constraints are violated, the barrier function will approach to infinity. There are many functions that can be served as the barrier function [14]–[16]. Here, to characterize such a class of barrier functions, the following assumption is used.

For ease of representation, consider the non-dimensional positive parameter $\widetilde{e} \triangleq \dfrac{\mathbf{e}^\intercal\mathbf{e}}{\varepsilon_b^2}$ in the rest of the paper.

Assumption 4: There exists a continuous differentiable barrier-Lyapunov-function (BLF), $V : [0,1] \to \mathcal{R}_{\geq 0}$ such that the following properties hold:

$$\begin{aligned}\alpha_1(\widetilde{e}) \leq V(\widetilde{e}) \leq \alpha_2(\widetilde{e}),\\ V(0) = 0; \qquad V(1) = \infty,\\ \left|\frac{\partial V}{\partial \mathbf{e}}(\widetilde{e})\right| \leq \alpha_3(\widetilde{e}),\\ \left[\frac{\partial V}{\partial \mathbf{e}}(\widetilde{e})\right]^\intercal P \frac{\partial V}{\partial \mathbf{e}}(\widetilde{e}) \geq \alpha_4(\widetilde{e}).\end{aligned} \tag{8}$$

where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ and $\alpha_4(\cdot)$ are class-$\mathcal{K}$ functions and $P = P^{\intercal} > 0$. □

Remark 6: If this BLF is always bounded for any $t \in [0, T_f]$, for any iteration, this indicates that output constraints are satisfied. The role of the proposed output feedback control law ensures that if the initial condition is within the domain of the attraction, this barrier-Lyapunov function will be always bounded for any $t \geq 0$. ◦

Remark 7: Many barrier functions in literature satisfy this assumption. For example, log-type barrier function from [22] and tan-type barrier function from [14]. ◦

With the existence of such a BLF function, an output feedback controller is proposed to drive the output of the system from violating the output constraints

$$\mathbf{u}^{fb} = \frac{\partial V}{\partial \mathbf{e}} \left(\frac{\mathbf{e}^{\intercal}\mathbf{e}}{\varepsilon_b^2} \right) \tag{9}$$

Remark 8: As the perfect tracking performance can be achieved by using a suitable ILC algorithm, the role of proposed output feedback law is to ensure that output constraints are satisfied with possibly bad tracking performance in time domain. When a better tracking performance is needed in the transient response in iteration domain, an extra output regulation control law, $\Psi(\mathbf{e})$, can be incorporated into the feedback loop. For example, the feedback loop can take the form of $\mathbf{u}^{fb} = \Psi(\mathbf{e}) + \left[\frac{\partial V}{\partial \mathbf{e}}(\mathbf{e}, \varepsilon_b)\right]$ to improve the tracking performance in time domain. ◦

Proposition 1 shows that the output constraints are satisfied when the proposed feedback (9) is used.

Proposition 1: Assume that Assumption 4 holds. For the system (7) with zero initial state and control input coming from (5) and (9), the output will satisfy $|\mathbf{e}(t)| \leq \varepsilon_b$, for all $t \in [0,\ T_f]$ when $\mathbf{u}^{ff} = 0$.

Proof: Taking the time derivative of BLF V along the trajectories of (7) yields,

$$\begin{aligned}\dot{V} &= \left[\frac{\partial V}{\partial \mathbf{e}}(\widetilde{e})\right]^{\intercal} \dot{\mathbf{e}} \\ &= \left[\frac{\partial V}{\partial \mathbf{e}}(\widetilde{e})\right]^{\intercal} \left(\left(\dot{C} + CA\right) \delta\mathbf{x} + CB\mathbf{u}_r - CB\frac{\partial V}{\partial \mathbf{e}}(\widetilde{e}) \right) \\ &\quad - \left[\frac{\partial V}{\partial \mathbf{e}}(\widetilde{e})\right]^{\intercal} CB \left(sat\left(\left[\frac{\partial V}{\partial \mathbf{e}}(\widetilde{e})\right], \mathbf{u}^* \right) - \frac{\partial V}{\partial \mathbf{e}}(\widetilde{e}) \right).\end{aligned} \tag{10}$$

As the saturation function satisfies global Lipschitz continuity condition, there exists a positive constant L_1 such that

$$\left| sat\left(\left[\frac{\partial V}{\partial \mathbf{e}}(\widetilde{e})\right], \mathbf{u}^* \right) - \frac{\partial V}{\partial \mathbf{e}}(\widetilde{e}) \right| \leq L_1 \left| \frac{\partial V}{\partial \mathbf{e}}(\widetilde{e}) \right| \tag{11}$$

Based on the Assumption 3, (10) can be re-written as

$$\dot{V} \leq C_{ur}\alpha_3(\widetilde{e}) - (1 - L_1)\alpha_4(\widetilde{e}), \tag{12}$$

where $C_{ur} = C_u + C_r$, $C_r = \max_{t\in[0,\ T_f]} \left| \left(\dot{C} + CA\right) \delta\mathbf{x} \right|$, $C_u = \max_{t\in[0,\ T_f]} |CB\mathbf{u}_r|$. Hence there exists a domain of the attraction $\mathcal{D}$ such that $\dot{V} \leq 0$ for any $|\mathbf{e}| \leq \mathcal{D}$. Noticing that $V(0) = 0$ as $\widetilde{e}(0) = 0$ due to Assumption 2. Consequently, there exists a positive constant M such that $V(t) \leq M$ for all $t \in [0,\ T_f]$. Hence the output constraints are satisfied. This completes the proof. □

The proposed output feedback (9) can satisfy output constraints. But the tracking performance cannot be guaranteed.

Next shows that by incorporating the proposed output feedback with a feed-forward ILC law, the perfect tracking performance can be achieved without violating input and output constraints.

IV. MAIN RESULT

With both output feedback (9) and feed-forward ILC (6) and input constraints (5), a sufficient condition is obtained to ensure the convergence of the tracking error in the presence of input and output constraints.

Theorem 1: Assume that the system (3) satisfies the Assumptions 1–3. Assume that Assumption 4 holds. Then the closed loop system with control laws (5), (6) and (9) achieves

1) perfect tracking performance in the presence of input and output constraints such that $\lim_{i\to\infty} \mathbf{e}_i = 0$ uniformly;
2) uniform boundedness and $\mathcal{L}^2$ norm convergence of feed-forward control input, i.e. $\lim_{i\to\infty} \mathbf{u}_i^{ff} = \mathbf{u}_r$,

if the convergence condition: $|I_m - \Gamma CB| < 1$ is satisfied.

Proof: A barrier composite energy function (BCEF) E_i proposed in [15] is used in this paper for the proof of the theorem.

$$\begin{aligned}E_i(t) =& e^{-\lambda t} V_{i-1}(t) + \int_0^t e^{-\lambda\tau} \delta\mathbf{u}_i^{ff\intercal}(\tau)\delta\mathbf{u}_i^{ff}(\tau) d\tau \\ & \forall t \in [0, T_f], i \in \mathcal{N}, \lambda > 0,\ V_0(t) = 0,\end{aligned} \tag{13}$$

where $\delta\mathbf{u}_i^{ff} = \mathbf{u}_r - \mathbf{u}_i^{ff}$. In order to ensure the satisfaction of constraints in each iteration, the induction based proof technique similar to [15] is used in this proof. Noted that there exists a postive constant Δ_1 such $\|\mathbf{e}_i\|_s \leq \Delta_1$ for any iteration.

Firstly, E_1 is finite as $V_0 = 0$ and $\mathbf{u}_1^{ff} = 0$. By Proposition 1, the output constraints are satisfied.

Secondly, we will show that $E_{j+1} \leq E_j$. This will lead to the conclusion that $\mathbf{e}_{j+1}$ satisfy output constraints.

The difference in BCEF between two consecutive iterations is given by $\Delta E_{j+1} = E_{j+1} - E_j$. From (13), it follows:

$$\begin{aligned}\Delta E_{j+1} =& e^{-\lambda t} \left(V_j - V_{j-1}\right) \\ & + \int_0^t e^{-\lambda\tau} \left(\left|\delta\mathbf{u}_{j+1}^{ff}\right|^2 - \left|\delta\mathbf{u}_j^{ff}\right|^2 \right) d\tau.\end{aligned} \tag{14}$$

Let us first establish a few relations that is useful for the proof of theorem. The first term in (14) can be written as:

$$e^{-\lambda t} V_j = \int_0^t e^{-\lambda\tau} \dot{V}_j d\tau - \lambda \int_0^t e^{-\lambda\tau} V_j d\tau. \tag{15}$$

But $\dot{V}_j$ can be written as:

$$\begin{aligned}\dot{V}_j &= \left[\frac{\partial V_j}{\partial \mathbf{e}}\right]^{\intercal} \left(\left(\dot{C} + CA\right) \delta\mathbf{x}_j + CB\delta\mathbf{u}_j \right) \\ &\leq C_d \alpha_3(\widetilde{e}) |\delta\mathbf{x}_j| + |CB| \alpha_3(\widetilde{e}) |\delta\mathbf{u}_j|.\end{aligned} \tag{16}$$

where $\delta\mathbf{u}_j = \mathbf{u}_r - \mathbf{u}_j$, $C_d = \max_{t\in[0,T_f]} \left|\dot{C} + CA\right|$.

The solution of (7) is given by

$$\delta\mathbf{x}_j = \Phi(t, 0)\delta\mathbf{x}_j(0) + \int_0^t \Phi(t, \tau)B(\tau)\delta\mathbf{u}_j(\tau)d\tau, \tag{17}$$

where $\Phi(t, \tau)$ is the state transition matrix.

However, as per Assumption 2, $\delta\mathbf{x}_j(0)=0$. Using Lemma 2 on $|\delta\mathbf{u}_j|$ yields:

$$|\delta\mathbf{u}_j| = |\mathbf{u}_r - sat(\mathbf{v}_j,\mathbf{u}^*)| = |\mathbf{u}_r - \mathbf{v}_j - (sat(\mathbf{v}_j,\mathbf{u}^*) - \mathbf{v}_j)| \\ \le \left|\delta\widetilde{\mathbf{u}}_j^{ff}\right| + 2\alpha_3(\widetilde{e}_j). \tag{18}$$

Taking the vector norm on each sides of (17) and using (18) yields:

$$|\delta\mathbf{x}_j| \le C_a\,|\delta\mathbf{u}_j| \le C_a\left|\delta\widetilde{\mathbf{u}}_j^{ff}\right| + 2C_a\alpha_3(\widetilde{e}_j), \tag{19}$$

where $C_a = |B|\int_0^t |\Phi(t,\tau)|\,d\tau$.

Using completion of squares, there exists a $\beta>0$ s.t.

$$\alpha_3(\widetilde{e}_j)\left|\delta\widetilde{\mathbf{u}}_j^{ff}\right| \le \frac{\beta}{2}\alpha_3^2(\widetilde{e}_j) + \frac{1}{2\beta}\left|\delta\widetilde{\mathbf{u}}_j^{ff}\right|^2. \tag{20}$$

Therefore, using (16), (19) and (20), there exists two positive constants ν_1 and ν_2 such that (15) can be written in the form:

$$e^{-\lambda t}V_j \le -\lambda\int_0^t e^{-\lambda\tau}\alpha_1(\widetilde{e}_j)d\tau + \nu_1\int_0^t e^{-\lambda\tau}\alpha_3^2(\widetilde{e}_j)d\tau \\ + \nu_2\int_0^t e^{-\lambda\tau}\delta\widetilde{\mathbf{u}}_j^{ff}d\tau. \tag{21}$$

Substituting (7) into (6) yields:

$$\delta\mathbf{u}_{j+1}^{ff} = \delta\widetilde{\mathbf{u}}_j^{ff} - \Gamma\dot{\mathbf{e}}_i = \mathrm{P}\delta\widetilde{\mathbf{u}}_j^{ff} + \mathbf{z}_j, \tag{22}$$

where $\mathrm{P} = (I_m - \Gamma CB)$ and $\mathbf{z}_j = -\Gamma\left(\dot{C} + CA\right)\delta\mathbf{x}_j + \Gamma CB(\delta\widetilde{\mathbf{u}}_j^{ff} - \delta\mathbf{u}_j)$.

Using Lemma 2 on $\left|\delta\widetilde{\mathbf{u}}_j^{ff} - \delta\mathbf{u}_j\right|$ yields

$$\left|\delta\widetilde{\mathbf{u}}_j^{ff} - \delta\mathbf{u}_j\right| = \left|\mathbf{u}_j - \widetilde{\mathbf{u}}_j^{ff}\right| = \left|sat(\mathbf{v}_j,\mathbf{u}^*) - \mathbf{v}_j + \mathbf{u}_j^{fb}\right| \\ \le 2\left|\mathbf{u}_j^{fb}\right| = 2\alpha_3(\widetilde{e}_j). \tag{23}$$

Using (23) and (19), it is further possible to show that there exists constants C_s, C_t such that

$$|\mathbf{z}_j| \le C_s\left|\delta\widetilde{\mathbf{u}}_j^{ff}\right| + C_t\alpha_3(\widetilde{e}_j). \tag{24}$$

Applying Lemma 1, followed by substituting (22) in the second term of difference of BCEF yields:

$$\left|\delta\mathbf{u}_{j+1}^{ff}\right|^2 - \left|\delta\mathbf{u}_j^{ff}\right|^2 \le \delta\mathbf{u}_{j+1}^{ff\,\intercal}\delta\mathbf{u}_{j+1}^{ff} - \delta\widetilde{\mathbf{u}}_j^{ff\,\intercal}\delta\widetilde{\mathbf{u}}_j^{ff} \\ = -\delta\widetilde{\mathbf{u}}_j^{ff\,\intercal}(I_m - \mathrm{P}^\intercal\mathrm{P})\delta\widetilde{\mathbf{u}}_j^{ff} + |\mathbf{z}_j|^2 + 2\mathbf{z}_j^\intercal\mathrm{P}\delta\widetilde{\mathbf{u}}_j^{ff} \\ \le -\lambda_p\left|\delta\widetilde{\mathbf{u}}_i^{ff}\right|^2 + |\mathbf{z}_i|^2 + 2\,|\mathbf{z}_i|\left|\mathrm{P}\delta\widetilde{\mathbf{u}}_i^{ff}\right|, \tag{25}$$

where $\lambda_p = \lambda_{max}(I_m - \mathrm{P}^\intercal\mathrm{P})$. If $|P|<1$, then $\lambda_p>0$. There exists a Γ for a given CB which satisfies this condition. This condition is same as the convergence condition from CM based analysis when D-type ILC is used without input saturation and output constraints.

Substituting for $\mathbf{z}_j$ from (24) in (25) and using completion of squares (20), it can be shown that there exists two positive constants: β_1 and β_2 such that (25) can be written as:

$$\left|\delta\mathbf{u}_{j+1}^{ff}\right|^2 - \left|\delta\mathbf{u}_j^{ff}\right|^2 \le -(\lambda_p - \beta_1)\left|\delta\widetilde{\mathbf{u}}_i^{ff}\right|^2 + \beta_2\alpha_3^2(\widetilde{e}_j). \tag{26}$$

Finally, substituting (26) and (21) into (14) yields:

$$\Delta E_{j+1} \le -\lambda\int_1^t e^{-\lambda\tau}\alpha_1(\widetilde{e}_j)d\tau + \overline{\nu}_1\int_0^t e^{-\lambda\tau}\alpha_3^2(\widetilde{e}_j)d\tau \\ - (\lambda_p - \overline{\nu}_2)\int_0^t e^{-\lambda\tau}\delta\widetilde{\mathbf{u}}_j^{ff}d\tau, \tag{27}$$

where $\overline{\nu}_1 = \nu_1 + \beta_2$ and $\overline{\nu}_2 = \beta_1 + \nu_2$. It is therefore possible to find a $\lambda>0$ and $\lambda_p > \overline{\nu}_2$ for some $\beta>0$ such that $\Delta E_{j+1} \le 0$. This leads to $\mathbf{e}_j \in \mathcal{D}$. So by induction, for any $j \in \mathcal{N}$, the constraints are satisfied.

Moreover, as E_j is non increasing along the iteration axis, it is possible to conclude that the tracking error and feed-forward control input uniformly converges, based on the similar conclusions in [11], [15]. This completes the proof. □

V. ILLUSTRATIVE EXAMPLES

For the illustration purpose, a *tan*-type barrier function proposed in [13] is used in this section which satisfy the properties given in Assumption 4. It has the following form

$$V = k\frac{\varepsilon_b^2}{\pi}\tan\left(\frac{\pi\mathbf{e}^\intercal\mathbf{e}}{2\varepsilon_b^2}\right), \tag{28}$$

where $k>0$ is a gain constant. This results in the following feedback control as per (9):

$$\mathbf{u}^{fb} = k\sec^2\left(\frac{\pi\mathbf{e}^\intercal\mathbf{e}}{2\varepsilon_b^2}\right)\mathbf{e}. \tag{29}$$

Two illustrative examples are given in this section. The first one is a single-input-single-output system from the motivating example given in this paper. The second one is a hypothetical MIMO LTV system.

A. Revisiting the motivating example

A feedback gain $k=0.01$ is chosen to demonstrate the effectiveness of proposed feedback. A saturation limit of $u^*=1$ is selected. The simulation results with the proposed feedback, shown in Fig. 3, indicates that the output constraints are satisfied in all iterations without violating the input constraints.

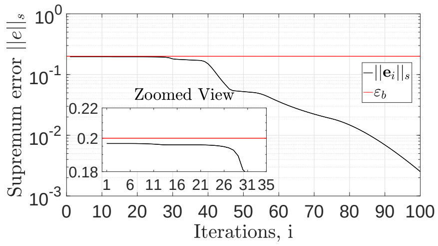


Fig. 3. Supremum norm from motivating example with proposed feedback

B. MIMO LTV Model

Next a MIMO system is considered, i.e., an LTV system (3), with state matrices $A = \begin{bmatrix} 0 & 1 & \mathrm{e}^{-0.5t} & 0 \\ 0 & -2 & 0 & -1.5 \\ 0 & \mathrm{e}^{-0.5t} & 0 & 1 \\ 0 & -0.75 & \sin^2(t) & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 2.5+\sin^2(t) & 0.45+\mathrm{e}^{-0.5t} \\ 0 & 0 \\ 0.45+\mathrm{e}^{-0.5t} & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1+\mathrm{e}^{-2t} & 0 & 0 \\ 0 & 0 & 0 & 1+\mathrm{e}^{-2t} \end{bmatrix}$. Let the output reference trajectory be $\mathbf{y}_r(t) =$

$\left[0.4\sin^3\left(\pi\frac{t}{T_f}\right), \quad 0.5\sin^3\left(\pi\frac{t}{T_f}\right)\cos\left(\pi\frac{t}{T_f}\right)\right]^\intercal$. The error bound $\varepsilon_b = 0.2$ and input saturation $\mathbf{u}^* = 1$ are chosen for simulation.

For the feedback control, the gain constant $k = 0.2$ is selected and the finite time $[0, 3]$ is considered. For the design of ILC algorithm, a constant feedback gain $\Gamma = \begin{bmatrix} 0.247 & -0.037 \\ -0.037 & 0.206 \end{bmatrix}$ is chosen, which satisfies the convergence condition: $|I_2 - \Gamma C(t)B(t)| \leq 1 \ \forall t \in [0, 3]$. The simulation is performed for 50 iterations. The variation of supremum norm of tracking error $\|\mathbf{e}_i\|_s$ is shown in Fig. 4, which demonstrates the convergence of tracking error. A zoomed view is provided to clarify that the output constraints are not violated in any iteration during the transient stages of learning. The output trajectories for a few iterations 1,5,10,15 and 50 for $y^1(t)$ are shown in Fig. 5, which clearly demonstrates that the trajectories are constrained within an ε_b bound with respect to reference trajectory. The control input signals of those iterations are shown in Fig.5. It indicates that both the input constraints and the output constraints are satisfied.

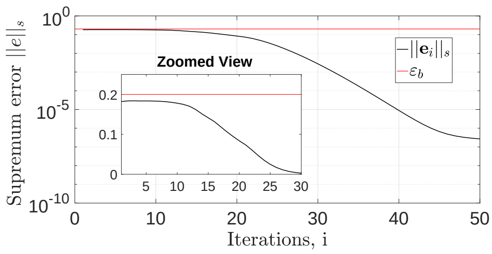


Fig. 4. Supremum norm-MIMO LTV model

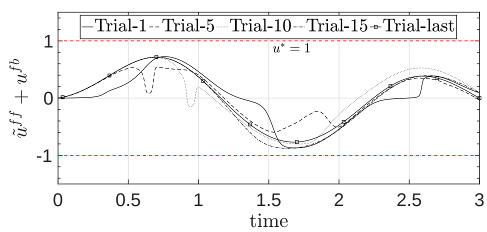


Fig. 5. Total Input, $v = \tilde{u}^{ff} + u^{fb}$ for control input-1, MIMO LTV

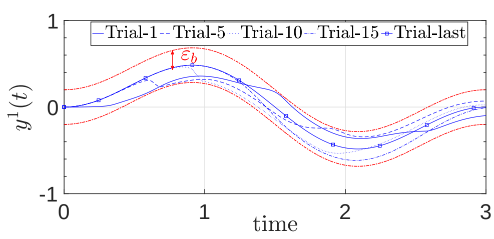


Fig. 6. Output Trajectory, $y^1(t)$ from MIMO LTV

VI. Conclusion

This paper proposed an output feedback-based ILC scheme to satisfy both the input and output constraints for a linear-time-varying system. In the proposed control scheme, the output feedback design is based on barrier Lyapunov function and the ILC is based on contraction mapping based approach. With the help of a barrier composite energy function, the proposed control law can ensure the perfect tracking performance in iteration domain without violating the input and output constraints in time-domain. The future work includes extending this control framework to a wider class of nonlinear systems.

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