Convergence of a function

Instructor's Notes

Introduction

This applet illustrates the ϵ-δ definition of the convergence and continuity of a function at a point, seen in real analysis courses. The formal definition goes as follows: a function $f : (a, b) \to \mathbb{R}$ converges to a real number L as $x \to x_0 \in [a, b]$ if, for all $\epsilon > 0$, there is some $\delta > 0$ such that $0 < |x-x_0| < \delta$ implies that $|f(x)-L| < \epsilon$. This is written $\lim_{x\to x_0} f(x) = L$. We also recall that $f(x)$ is continuous at $x_0 \in (a, b)$ if $\lim_{x\to x_0} f(x) = f(x_0)$.

About the applet

General description and functionalities

In the applet, there are two modes: a **convergence** mode and a **continuity** mode. For both modes, the parameter in the definition are visually indicated and controllable. You can input your own function $f(x)$ and set the candidate limit L in convergence mode. You can turn ϵ and δ on and off, and manually determine the ϵ "envelope" around L or $f(x_0)$ and the δ "envelope" around x_0. More detail on each mode:

- In **convergence mode**, when both ϵ and δ are on, "bad" segments of the graph of $f(x)$ which satisfy $0 < |x - x_0| < \delta$ but not $|f(x) - L| < \epsilon$ are labelled red. "Good" points which satisfy $0 < |x-x_0| < \delta$ and $|f(x)-L| < \epsilon$ are labelled green. "Irrelevant" segments of the graph with $|x-x_0| \geq \delta$ are gray. The formal definition is at the bottom of the page and colour coded to match the interactive applet elements.
- **Continuity mode** is the similar to convergence mode, but the limit $L = f(x_0)$ is fixed and the point on the graph $(x_0, f(x_0))$ is clearly indicated by a green circle.

Notation and colour coding

The notation and colour coding in the applet have been designed to reinforce each other. The formal definition of ϵ-δ convergence is displayed at the bottom of the applet, with different

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in E \quad \left(0 < |x - x_0| < \delta \Longrightarrow |f(x) - L| < \varepsilon\right)$$

Figure 1: Notation

parts coloured to match the corresponding applet elements. The notation is consistent with courses run by the School of Mathematics and Statistics at the University of Melbourne.

Instructions for using the applet

Enter a rule for your function in the box labelled `fn`. There should be a single independent variable `x`. Geogebra recognises elementary functions like `sin`, `cos`, `sqrt`, and `ln`. A complete list can be found here. We have also built in an approximation of the *Dirichlet function* $d(x)$ defined by

$$d(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

It may be invoked with `d(x)`.[1]

Select a mode (convergence or continuity) using the boxes on the bottom right. Drag the purple circle •, or the connected horizontal line, to change the point x_0 at which we are evaluating the limit. In **convergence mode**, the brown circle • indicates the limit L of the function at x_0. Drag the circle, or the connected horizontal line, to change L. In **continuity mode**, the limit is automatically set to $f(x_0)$. The ϵ enveloped around the limit may be toggled on and off. It can be controlled by dragging the orange circle • on the vertical axis or moving the horizontal boundaries. The (deleted) δ neighbourhood around x_0 can also be toggled,. The neighbourhood can be controlled by dragging the blue circle • or either vertical boundary line. When both ϵ and δ are on

- "irrelevant" segments of the graph of $f(x)$ which do not lie in the envelope are gray;
- "bad" segments which lie in the δ envelope but not the ϵ envelope are red;

[1] The Dirichlet function $d(x)$ is approximated as the ninth binary digit of x, which varies rapidly enough for illustrative purposes.

- "good" segments which lie in both the δ and ϵ envelope are green. In **continuity mode**, the point $(x_0, f(x_0))$ is clearly indicated by a green circle •.

You can zoom using the scroll wheel or `+/t` buttons at the top of the page. If you hold `Shift`, you can click and drag both axes.

Example usage

Here are some good examples which have been tested on the applet.

1. Enter **convergence** mode. Enter `f(x)=x^2` into the function input box, set $x_0 = 1$ and $L = 1$. If you decrease ϵ, you must decrease δ to ensure that there are no red segments. A short proof shows that $\delta < 2\epsilon$ will always work for $\epsilon < 1$, and this can be verified with the applet. This is a simple example of convergence, and also shows how the horizontal envelope width δ depends on ϵ.

2. Enter **convergence** mode. In the example above, set $L = 1.5$ and $\epsilon = 0.5$. No matter the choice of neighbourhood width δ, the graph will always have red segments. This is an example of a function not converging to L.

3. Enter **convergence** mode. Enter $f(x) = d(x)$, the Dirichlet function. Set $x_0 = 1$, L to any value and $\epsilon = 0.5$. It should be clear that no choice of δ will eliminate red segments of the graph. Since this is true for any value of L, we have a simple example of limit not existing.

4. Enter **continuity** mode. The first example, $f(x) = x^2$, $x_0 = 1$ and $L = 1$, gives an example of a function which is continuous at x_0, since $L = f(x_0)$. Visually, this is indicated by the green dot at $(1, 1)$.

5. Enter **continuity** mode. Input `f(x)=If[0.99<x<1.01,2,x]` to simulate a function with a discontinuity at 1. Set $x_0 = 1$ and $L = 1$. Ignoring the red point at $(1, 2)$, we see that, for any choice of ϵ, setting $\delta = \epsilon$ clears away red segments. Hence, the limit is 1. The red point indicates that the limit of the function at $x_0 = 1$ and the value of the function $f(x_0)$ disagree, so f is discontinuous there.

Suggestions for using the applet

The applet can be used to illustrate

- the dependence of the horizontal envelope width δ on the vertical envelope width ϵ;
- why, in terms of ϵ-δ, a function converges to L at x_0;
- why, in terms of ϵ-δ, a function does not converge to L at x_0;
- why, in terms of ϵ-δ, a limit may not exist at x_0;
- why, in terms of ϵ-δ, a function is continuous at x_0;
- why, in terms of ϵ-δ, a function is discontinuous at x_0.

Some possible classroom uses for the applet:

- as a lecture demonstration to accompany the formal definition;
- in computer labs, where students can explore the definition interactively;
- in assignments, or as part of computer-based assessment;
- for students to use outside of class as a learning tool;
- in one-on-one consultation.