Arithmetic Jets

X smooth / field k

$$J_{geom}(X) = \lim_{r \to \infty} X\left(\frac{R[t]}{(t^r)}\right)$$

R a ring $\rightsquigarrow$ 2 universal δ-rings

$$W(R) = \varprojlim_n W_n(R)$$

$$J(R) = \operatorname{colim}_n J_n(R)$$

$$X : \mathrm{Rings} \longrightarrow \mathrm{Set}, \mathrm{Grpd}$$

$$W(X) = \operatorname{colim}_n W_n(X), \quad W(\operatorname{Spec} R) = \operatorname{colim}_n \operatorname{Spec}(W_n(R))$$

$$J(X) = \lim_n J_n(X), \quad J(\operatorname{Spec} R) = \lim_n \operatorname{Spec}(J_n(R)) = \operatorname{Spec}(J(R)).$$

$$\Rightarrow \quad J(X)(R) = \varprojlim_n X(W_n(R))$$

φ

lift of frobenius

<u>E.g.</u> $J(\mathbb{A}^1) = \mathrm{Spec}(\mathbb{Z}[x_0, x_1, \ldots])$

$$= \mathrm{Spec}(\mathbb{Z}\{x\})$$

$$[\mathbb{A}^1/\mathbb{G}_m](R) = \{\eta : \underset{\substack{\uparrow \\ \text{rk } 1,\ \text{proj } R\text{-mod}}}{L} \to R\}$$

$$J([\mathbb{A}^1/\mathbb{G}_m])(R) = \varprojlim_n \{\eta_n : L_n \to W_n\}$$

if p nilpotent $\simeq \{\eta : L \to W(R)\}$

$$\therefore \quad \Sigma \hookrightarrow J([\mathbb{A}^1/\mathbb{G}_m])$$

$$\Sigma = \{ s : \underset{\substack{\text{dist} \\ \text{q. ideal}}}{I} \to W(R) \}$$

this is fully faithful. ($\Sigma(R)$ is $\emptyset$ if p <u>not</u> nilpotent on R)

essential image is those $\eta : L \to R$ s.t. $\exists n > 0$ with

$$\varphi^{*n}(\eta) \simeq (W(R) \xrightarrow{p} W(R))$$

PRISMATIZATION

$X / \mathrm{Spf}(\mathbb{Z}_p)$ smooth qcqs p-adic formal scheme

X^{Δ} / Σ a qcqs flat alg. stack over Σ

$$\left[J_2^{\Delta}(X) \rightrightarrows J_1^{\Delta}(X) \right] \xrightarrow{\sim} X^{\Delta}$$

$$J_1^{\Delta}(X) = J(X) \times \Sigma$$

$$\Rightarrow J_1^{\Delta}(X)(R, \xi : I \to W(R)) = J(X)(R)$$

$$J_2^{\Delta}(X)(R, \xi : I \xrightarrow[\text{q.i.}]{\text{dist.}} W(R))$$

$\uparrow$

Σ-rings

$$W_{\xi}^2(R) = \{ (r, r', z) \in W(R)^2 \times I : r - r' = \xi(z) \}$$

$$= \{ (r, z) \in W(R) \times I \}$$

addition is coordinatewise

multiplication: $(r, z) \cdot (s, w) = (rs, rw + sz + \xi(z)w)$

$\rightsquigarrow$

$$W^2_\xi(R) \;\underset{r}{\overset{r'}{\substack{\longrightarrow\\ \longleftarrow \\ \longrightarrow}}}\; W(R)$$

"graded object in $W(R)$-algebras"

$\rightsquigarrow$

$$\left[\mathcal{J}^{\Delta}_2(X)(\xi : I \to W(R)) \;\substack{\longrightarrow\\ \longleftarrow \\ \longrightarrow}\; \mathcal{J}^{\Delta}_1(X)(\;"\;) \right]$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$\left[\varprojlim_n X(W^2_{\xi,n}(R)) \;\substack{\longrightarrow\\ \longleftarrow \\ \longrightarrow}\; \varprojlim_n X(W_n(R)) \right]$$

$$\shortparallel$$

$$X^{\Delta}(R, \xi : I \to W(R))$$