What is Σ (continued)?

$$[\mathbb{W}_{dist}/\mathbb{W}^*] \xrightarrow{\sim} \Sigma \xrightarrow{h} [\hat{\mathbb{A}}^1/\mathbb{G}_m] \hat{\otimes} \mathbb{Z}_p$$

$$\overset{\delta}{\circlearrowleft} \mathbb{W} \overset{\varphi}{\circlearrowright} = \operatorname{colim}_n \operatorname{Spec}(\mathbb{Z}/p^n[x_0, x_1, \ldots]), \quad \delta(x_i) = x_{i+1}$$

$$\varphi(x_i) = x_i^p + px_{i+1}$$

$$\cup$$

$$\mathbb{W}^* = \operatorname{colim}_n \operatorname{Spec}(\mathbb{Z}/p^n[x_0^{\pm}, x_1, \ldots])$$

φ is affine, faithfully flat

Flat affine group schemes over $\operatorname{Spf}(\mathbb{Z}_p) = \operatorname{colim}_n \operatorname{Spec}(\mathbb{Z}/p^n)$

$$\mathbb{W} \supseteq \mathbb{W}_{dist} = \operatorname{colim}_{n,m} \operatorname{Spec}\left(\mathbb{Z}/p^n[x_0, x_1^{\pm}, \ldots]/(x_0^m)\right)$$

$$\underset{\substack{\| \\ (\xi_0, \xi_1, \ldots)}}{\xi} \in \mathbb{W}_{dist} \iff \xi_0 \text{ nilpotent}, \ \delta(\xi) = (\xi_1, \xi_2, \ldots) \text{ is a unit} \iff \xi \text{ is a unit}$$

① $\Delta \subseteq \Sigma$ describe this

② Describe fiber products over Σ:

$$\operatorname{Spec}(R) \times_{\Sigma} \operatorname{Spec}(R')$$

affine + faithfully flat

$$\begin{tikzcd}
\xi \arrow[r, mapsto] & \xi_0 \in \\
\mathbb{W}_{dist} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxtimes"] & \hat{A}' \hat{\otimes} \mathbb{Z}_p \arrow[d] \\
\Sigma \arrow[r, "h"] & {[\hat{A}'/\mathbb{G}_n] \hat{\otimes} \mathbb{Z}_p} \\
\Delta \arrow[r] \arrow[u, hook] \arrow[ur, phantom, "\boxtimes"] & {[0/\mathbb{G}_n] \hat{\otimes} \mathbb{Z}_p} \arrow[u, hook]
\end{tikzcd}$$

$[\hat{A}'/\mathbb{G}_n] \hat{\otimes} \mathbb{Z}_p = \left(\eta : L \to R \text{ s.t. } \exists n \text{ w/ } \eta^{\circ n} = 0 \right)$

$[0/\mathbb{G}_n] \hat{\otimes} \mathbb{Z}_p = (\eta = 0)$

$$\Delta(R) = \{ \underset{\text{distinguished}}{\xi : I \to W(R)} : \xi(I) \subseteq VW(R) = \ker(W(R) \to R) \}$$

II) $V : \varphi_* W(R) \xrightarrow{\sim} VW(R) = \ker(W(R) \to R)$

$$V(\varphi(w_1) w_2) = w_1 V(w_2)$$

I) a) $Vw \in VW(R)$ is distinguished iff $w \in W(R)^*$

b) $\xi : I \to VW(R) \subseteq W(R)$

$$\Rightarrow \Delta(R) = \{ \xi : I \to VW(R) \subseteq W(R) \text{ distinguished} \}$$

$$= \{ V^{-1} \circ \xi : I \to \varphi_* W(R) \text{ condition} \}$$

$$= \{ \tilde{\xi} : \varphi^* I \xrightarrow{\sim} W(R) \}$$

$$= \left(\mathrm{Spf}(\mathbb{Z}_p) / \mathbb{W}^*[\varphi] \right)(R)$$

where

$$\mathbb{W}^*[\varphi] = \ker(\mathbb{W}^* \xrightarrow{\varphi} \mathbb{W}^*)$$

$$\left[\begin{array}{l} \mathbb{G}_m\text{-torsor over } \mathrm{Spec}(R) \simeq \text{projective rk 1 } R\text{-mod} \\ \mathbb{W}^*\text{-torsor over } \mathrm{Spec}(R) \simeq \text{proj rk 1 } W(R)\text{-mod} \end{array} \right]$$

$\mathbb{W}^*[\varphi]$-torsor :

$$\mathrm{eq}\left(\underline{\mathrm{Isom}}_{\mathbb{W}}(I \otimes_{W(R)} \mathbb{W}, \mathbb{W}) \underset{\xi}{\overset{\varphi}{\rightrightarrows}} \underline{\mathrm{Isom}}_{\mathbb{W}}(\varphi^* I \otimes \mathbb{W}, \mathbb{W}) \right)$$

What is $\mathbb{W}^*[\varphi] \simeq \mathbb{W}[\varphi] = \ker(\mathbb{W} \xrightarrow{\varphi} \mathbb{W})$

$$w \longmapsto 1-w$$

$$\mathbb{W}[\varphi](R) = \{ (x_0, x_1, \dots) : x_0^p + p x_1 = 0,\ x_1^p + p x_2 = 0, \dots$$

$$\simeq \{ (y_0, y_1, \dots) : y_0^p = p y_1,\ y_1^p = p y_2, \dots \}$$

$$\simeq \mathrm{Hom}\left(\mathbb{Z}_p\left[y, \frac{y^p}{p}, \frac{y^{p^2}}{p^{1+p}}, \frac{y^{p^3}}{p^{1+p+p^2}}, \dots\right], R \right)$$

subring of $\mathbb{Q}_p[y]$.

$$\simeq \mathrm{Hom}\left(\mathbb{Z}_p\left[y, \frac{y^2}{2!}, \frac{y^3}{3!}, \dots\right], R \right)$$

DIVIDED POWER RING

$$W^*[\eta] \xrightarrow{\sim} \mathbb{W}[\eta] \simeq \operatorname*{colim}_n \operatorname{Spec}\left(\mathbb{Z}/p^n\left[\tfrac{y^i}{i!} : i \geq 0\right]\right)$$

W^* is flat but not of finite type

LEMMA

$$\begin{tikzcd} \mathcal{X} \arrow[r, "g"] \arrow[dr, "f"'] & \Sigma \arrow[d, "h"] \\ & {[\hat{\mathbb{A}}^1/\mathbb{G}_m] \hat{\otimes} \mathbb{Z}_p} \end{tikzcd}$$

If f is algebraic + flat, then g is algebraic + flat & f, g are faithfully flat.

FIBER PRODUCTS OVER Σ

δ ring A, $A\{x\} = A[x_0, x_1, \ldots]$

$$\delta(x_i) = x_{i+1}$$

$$\operatorname{Hom}_{\delta\text{-}A}(A\{x\}, B) = B$$

$I \subseteq A$, I_δ smallest δ-stable ideal.

$$I = (f_j : j \in J) \Rightarrow I_\delta = (f_j, \delta(f_j), \delta^2(f_j), \ldots : j \in J).$$

$$A\{x^{\pm 1}\} = \frac{A\{x, y\}}{(xy-1)_\delta} \xRightarrow{\hat{p}} A[x_0^{\pm 1}, x_1, \ldots]^{\hat{p}}$$

$$
\begin{tikzcd}
\mathrm{Spec}(R) \arrow[r, "(W(R) \xrightarrow{\xi} W(R))"] & \Sigma \\
X \arrow[u] \arrow[r] \arrow[ur, phantom, "\boxtimes"] & \mathrm{Spec}(R') \arrow[u, "(W(R') \xrightarrow{\xi'} W(R'))"']
\end{tikzcd}
$$

$$X(S) = \{ f : R \to S,\ g : R' \to S,\ u \in W(S)^{\times} \text{ such that } W(f)(\xi) = u\, W(g)(\xi') \}$$

$$= \{ h : W(R) \otimes W(R') \to W(S),\ u \in W(S)^{\times} \text{ s.t. } h(VW(R) \otimes 1 + 1 \otimes VW(R')) \subseteq VW(S) \quad (*)$$
$$\& \quad h(\xi \otimes 1) = u \cdot h(1 \otimes \xi') \}$$

$$= \left\{ \tilde{h} : \frac{W(R) \otimes W(R')\{u^{\pm 1}\}}{(\xi \otimes 1 - u(1 \otimes \xi'))} \longrightarrow W(S) \right\} + (*)$$

$$= \mathrm{Hom}\left(\frac{W(R) \otimes W(R')\{u^{\pm 1}\}}{(\xi \otimes 1 - u(1 \otimes \xi'))_{\delta}} \otimes_{W(R) \otimes W(R')} (R \otimes R'),\ S \right)$$

$\therefore \quad X = \operatorname{Spec}(\qquad)$