Last time + ε

δ-rings: (A, δ) $\varphi(a) = a^p + p\delta(a)$ lifts Frob

Witt vectors: $W(R) \xrightarrow{pr} R$ universal δ-ring mapping to R

Teichmüller: $\begin{tikzcd} W(R)^{\delta=0} \subseteq W(R) \arrow[r] & R \end{tikzcd}$, with $W(R)^{\delta=0} \xrightarrow{\sim} R$, $[\ \ ]: R \longrightarrow W(R)$

Verschiebung: $V: W(R) \longrightarrow W(R)$ additive, injective, image $= \ker(pr)$

$$V(\varphi(w_1)w_2) = w_1 V(w_2), \quad \varphi(V(w)) = pw$$

$\Longrightarrow$ Every $w \in W(R)$ has

$$w = [r_0] + V[r_1] + V^2[r_2] + \dots$$
$$= \sum_{i=0}^{\infty} V^i[r_i]$$

If R is a perfect $\mathbb{F}_p$-algebra

$$V = p\varphi^{-1}, \quad w = \sum V^i[r_i] = \sum r_i^{1/p^i} p^i$$

* $V^n W(R) \subseteq W(R)$ ideal, $W_n(R) = W(R) / V^n W(R)$

$$W(R) = \varprojlim_n W_n(R)$$

p-adic rings

A topological ring A is p-adic if $A = \varprojlim_i A_i$, p nilpotent in A_i & $A_i \to A_j$ are surjective with nilpotent kernel.

[A_i are discretely topologized + A has limit topology]

(A, δ) is p-adic if δ is continuous.

e.g. $(\mathbb{Z}_p, \delta)$, $\mathbb{Z}_p = \lim \mathbb{Z}/p^n$

$$\mathbb{Z}_p[|t|] = \lim_{m,n} \left(\mathbb{Z}/p^n[t]\right)/t^m$$

If p nilpotent in R, then

$$W(R) = \varprojlim W_n(R)$$

Distinguished elements

(A, δ) p-adic. $\xi \in A$ is distinguished if

① ξ is topologically nilpotent

② $\delta(\xi)$ is a unit

These are stable under scaling by units & under continuous δ-maps e.g. (a) $\mathbb{Z}_p$, p is distinguished, $\delta(p) = \frac{p - p^p}{p} = 1 - p^{p-1}$

$a \in \mathbb{Z}_p$ distinguished $\iff$ $a = up$, $u \in \mathbb{Z}_p^\times$

(b) $(\mathbb{Z}_p[|t|], \delta(t) = 0)$, $\xi = p - t$ is distinguished

$$\delta(p-t) \equiv \delta(p) \mod (t)$$

Is $p^2 - t$ distinguished? NO! $\delta(p^2 - t) = \delta(p^2) \mod (t)$

Is $p - t^2$ distinguished? YES! $\delta(p - t^2) = \delta(p) \mod (t)$

$$\xi = \sum_{i=0}^{\infty} a_i t^i \qquad a_i \in \mathbb{Z}_p$$

$$= \sum_{i,j=0}^{\infty} s_j p^j t^i \qquad s_j \in \mu_{p-1} \cup 0 \in \mathbb{Z}_p$$

$$= \sum_{j=0}^{\infty} b_j(t) p^j \qquad b_j \text{ coeff's in } \mu_{p-1} \cup 0 \in \mathbb{Z}_p$$

Distinguished when $b_0(t)$ top. nilpotent & $b_1(t)$ is a unit.

$E(t)$ Eisenstein polynomial is distinguished

(c) $\xi \in W(R)$ distinguished if

$$\xi = \sum_{i=0}^{\infty} V^i [r_i]$$

$r_0 \in R$ nilpotent & r_1 is a unit

Σ

$\mathbb{W}$ = (pro) algebraic group of Witt vectors

$\cup\!\!|$

$\mathbb{W}^{\times}$ = " " " " units in Witt vectors

$$\mathbb{W} = \operatorname{Spec}(\mathbb{Z}[x_0, x_1, \dots])$$

$$\mathbb{W}^{\hat{p}} = \operatorname{colim}_n \operatorname{Spec}(\mathbb{Z}/p^n[x_0, x_1, \dots])$$

$\cup$

$\mathbb{W}^{*,\hat{p}}$

$\mathbb{W}_{dist} \subseteq \mathbb{W}^{\hat{p}}$ whose R-points are the distinguished elements of $W(R)$

i.e. $$\mathbb{W}_{dist}(R) = \begin{cases} \{\xi \in W(R) : \xi \text{ distinguished}\} & p \text{ nilpotent in } R \\ \emptyset & \text{otherwise} \end{cases}$$

$$\Rightarrow \mathbb{W}_{dist} = \operatorname{colim}_{n,m} \operatorname{Spec}(\mathbb{Z}/p^n[x_0, x_1^{\pm}, x_2, \dots]/(x_0^m))$$

$$\mathbb{W}^{*,\hat{p}} \curvearrowright \mathbb{W}_{dist}$$

$$\Sigma = \left[\mathbb{W}_{dist} / \mathbb{W}^{*,\hat{p}}\right]$$ fpqc stack (on rings) quotient

(A, δ) p-adic δ-ring.

A distinguished quasi-ideal $(\xi : I \to A)$ is:

① I is projective rk 1 A-module

② ξ is a linear map such that after localization & completion at any $f \in A$ that trivializes I, ξ sends a generator of I to a distinguished element.

Prop If p is nilpotent in R,

$$\Sigma(R) = \{\text{groupoid of distinguished quasi-ideals of } W(R)\}$$

Morphisms

$$\begin{tikzcd} I_1 \arrow[rr, "v"] \arrow[dr, "\xi_1"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowright"] & & I_2 \arrow[dl, "\xi_2"] \\ & A & \end{tikzcd}$$

(every such v is an isomorphism)

Properties of Σ

$\varphi : \Sigma \to \Sigma$, induced by Witt vector Frobenius.

$\Delta \subseteq \Sigma$ divisor, $\Delta(R) \subseteq \Sigma(R)$

$$\Delta(R) = \{ (\xi : I \to W(R)) : \xi(I) \subseteq \ker(pr) \}$$

If (A, δ),
$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr] & W(A) \arrow[d] \\ & A \end{tikzcd}$$
, if A is p-adic continuous

$(\xi : I \to A)$, base change along w to get an element of

$$\varprojlim \Sigma(A_i) \Longleftrightarrow \mathrm{Spf}(A) \longrightarrow \Sigma$$

with φ acting on $\mathrm{Spf}(A)$ and φ, Δ on Σ.

Aside

$$[\mathbb{A}^1 / \mathbb{G}_m](R) = \{ \text{groupoid of } \eta : L \to R,\ L \text{ proj. rk } 1 \}$$
$$\cup\!|$$
$$[\hat{\mathbb{A}}^1 / \mathbb{G}_m](R) = \{ \text{''} \quad \text{''} \quad \text{''} \quad \text{''} + \eta^{\otimes n} = 0 \text{ for some } n \}$$

$$h : \Sigma(R) \longrightarrow [\hat{\mathbb{A}}^1 / \mathbb{G}_m](R)$$
$$(\xi : I \to W(R)) \longmapsto (\xi \otimes_{W(R)} R : I \otimes_{W(R)} R \to R)$$

Prop h is algebraic & faithfully flat

[That is, the fibers of f are quotients of affine schemes by flat affine groupoids]

Breuil-Kisin cover $\delta(t)=0$.

$$\operatorname*{colim}_{m,n} \operatorname{Spec}\left(\mathbb{Z}/p^n[t]/t^m\right) = \operatorname{Spf}\left(\mathbb{Z}_p[[t]]\right) \xrightarrow{f_{BK}} \Sigma$$

Just need to specify object of:

$\Sigma\left(\mathbb{Z}/p^n[t]/t^m\right)$, so just need dist elt of $W\left(\mathbb{Z}/p^n[t]/t^m\right)$

Let $\xi = p-t : \mathbb{Z}_p[[t]] \xrightarrow{\delta} W(\mathbb{Z}_p[[t]]) \longrightarrow W\left(\mathbb{Z}/p^n[t]/t^m\right)$ $\xi_{m,n}$

Prop f_{BK} is affine & faithfully flat