

23/4 LANCE GURNEY

What is a prism?

Let k be a field. Too many good cohomology theories for varieties X/k .

ℓ -adic, $\ell \neq 0 \in k$

$$H^i_{\text{ét}}(X, \mathbb{Z}_\ell) \in \text{Rep}_{\mathbb{Z}_\ell}(\text{Gal}(\bar{k}/k))$$

algebraic de Rham

$$H^i(X, \Omega^{\bullet}_{X/k}) \in \text{Vect}^{\text{Fil}}_k(G_k)$$

Crystalline $p=0 \in k$, $H^i_{\text{crys}}(X/W(k)) \in \text{Mod}(W(k)[F])$

Betti, $k \hookrightarrow \mathbb{C}$

$$H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}) \in \text{Ab}$$

Relations

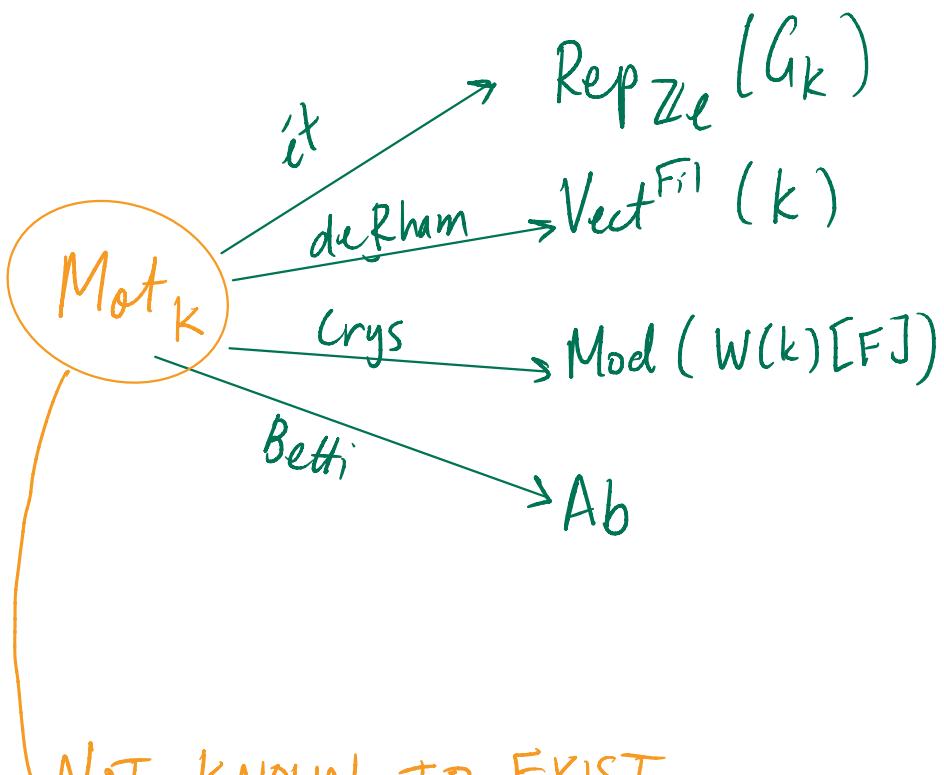
$$H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^i(X, \Omega^{\bullet}_{X/k}) \otimes_k \mathbb{C}$$

$$H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong H^i_{\text{ét}}(X, \mathbb{Z}_\ell)$$

Grothendieck

Motivic

$$h^i(X) \in \text{Mot}_k$$



Existence would follow from resolution of standard conjectures.

Goal Find an explicit cohomology theory that determines "all" the others.

Prismatic cohomology determines all known p-adic cohomology theories for p-adic varieties.

$$\begin{aligned}
 X/\mathbb{Z}_p &\quad \text{smooth \& proper} \\
 \text{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p}) &\ni H^i_{\text{ét}}(X_{\mathbb{Q}_p}, \mathbb{Z}_p) \\
 \text{Vect}^{\text{Fil}}(\mathbb{Z}_p) &\ni H^i(X, \Omega_{X/\mathbb{Z}_p}) \\
 \text{Mod}(\mathbb{Z}_p[F]) &\ni H^i_{\text{crys}}(X_{\mathbb{F}_p}/\mathbb{Z}_p)
 \end{aligned}$$

Drinfeld, Bhatt-Scholze

Prismatization functor:

$$\left\{ \begin{array}{l} p\text{-adic formal} \\ \text{schemes} \end{array} \right\} \xrightarrow{\quad} \left\{ \begin{array}{l} p\text{-adic stacks} \\ \text{over} \\ [\overset{\Delta}{A}/\mathcal{O}_m, \mathrm{Spf}(\mathbb{Z}_p)] \end{array} \right\}$$

$$X \xrightarrow{\quad} X^{\Delta} \quad \begin{matrix} \Delta \leftarrow \text{prism} \\ \leftarrow \text{prismatization} \end{matrix}$$

- * $\Sigma = \mathrm{Spf}(\mathbb{Z}_p)^{\Delta}$
- * $F: X^{\Delta} \rightarrow X^{\Delta}$ lift of Frobenius
- * $\Delta \subseteq \Sigma$ a divisor

Prismatic cohomology:

$$f^{\Delta}: X^{\Delta} \rightarrow \Sigma$$

$$H_{\Delta}^i(X) = R^i f_*^{\Delta} \mathcal{O}_{X^{\Delta}} \in Qcoh^{\mathrm{isog}}(\mathcal{O}_{\Sigma}[F])$$

$$\begin{array}{ccc} Qcoh^{\mathrm{isog}}(\mathcal{O}_{\Sigma}[F]) & \xrightarrow{\alpha} & \mathrm{Rep}_{\mathbb{Z}_p}(G_K) \\ \xrightarrow{dR} & & \\ & \searrow & \\ & \mathrm{Mod}(\mathbb{Z}_p[F]) & \end{array}$$

Topics

- ① Definitions / properties of prismatic realization & cohomology
- ② de Rham / crystalline realization
- ③ Étale realization (Kisin F-crystals & crystalline reps)
- ④ q-de Rham cohomology

What is a prism?

Σ is the moduli space of prisms (i.e. universal prism)

Definition A prism is a pair $(A, \mathfrak{F}: I \rightarrow A)$

- ① A is a δ -ring
- ② $\mathfrak{F}: I \rightarrow A$ is a distinguished quasi-ideal

A δ -ring is a pair (A, δ_A) , where $\delta_A: A \rightarrow A$

- ① $\delta(ab) = a^p \delta(b) + b^p \delta(a) + p \delta(a) \delta(b)$
- ② $\delta(a+b) = \delta(a) + \delta(b) - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} a^i b^{p-i}$
- ③ $\delta(1) = 0$

Point: $\varphi(a) = a^p + p\delta(a)$ is a ring homomorphism
 $\varphi(a) = a^p \bmod pA$ lifts Frobenius

① \mathbb{Z} $\delta(a) = \frac{a - a^p}{p}$, $\varphi(a) = a$
 $(\mathbb{Z}, \delta_{\mathbb{Z}}) \rightarrow (A, \delta_A)$ for any (A, δ_A) .

Follows from this that $A \ni p^i \neq 0 \quad \forall i \geq 1$.

② If A is p -torsion free, then

$$\varphi(a) \equiv a^p \pmod{pA}$$

$$\Leftrightarrow \delta(a) = \frac{\varphi(a) - a^p}{p}$$

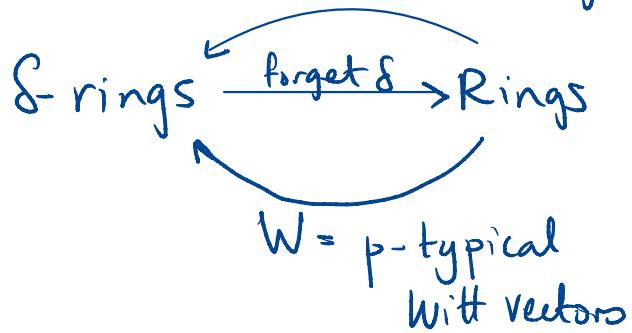
i.e. δ -ring \hookrightarrow lift of Frobenius

③ $\mathbb{Z}[t]$, $\varphi(t) = t^p$, $\delta(t) = 0$

④ $\mathbb{Z}[x]$, $\varphi(x) = x^p + px$, $\delta(x) = x$

⑤ $\mathbb{Z}[\zeta_n] \subseteq \mathbb{C} \quad (n, p) = 1$

Universal examples J = arithmetic jet space



1 $J(\mathbb{Z}[x]) = \mathbb{Z}\{x\} = \mathbb{Z}[x, \delta(x), \delta^2(x), \dots]$
free vars

$$\varphi(\delta^i(x)) = (\delta^i(x))^p + p\delta^{i+1}(x)$$

[2] J is a left adjoint, so if

$$R = \frac{\mathbb{Z}[x_i : i \in I]}{(f_j : j \in J)}$$

$$\Rightarrow J(R) = \frac{\mathbb{Z}[x_i, \delta^k(x_i); i \in I, k=1, \dots]}{(f_j, \delta(f_j), \delta^2(f_j), \dots : j \in J)}$$

[3] $W(R) = \text{Hom}_{\text{Rings}}(\mathbb{Z}\{\times\}, R) = \prod_{i=0}^{\infty} R$ as ab. groups

$$(r_0, r_1, \dots) + (s_0, s_1, \dots) = (r_0 + s_0, r_1 + s_1 - \sum_i^1 \binom{p}{i} r_0^{(p-i)} s_0^{p-i}, \dots)$$

[4] $W(\mathbb{F}_p) = \mathbb{Z}_p, W(\mathbb{F}_{p^r}) = \mathbb{Z}_p[\xi_{p^{r-1}}]$

[5] $W(\mathbb{F}_p[t, t^{\frac{1}{p}}, t^{\frac{1}{p^2}}, \dots]) \cong \mathbb{Z}_p[t, t^{\frac{1}{p}}, t^{\frac{1}{p^2}}, \dots]^p$

Analogy

$$\mathbb{Z}_p, (\wp)$$

\mathbb{Z}_p -algebra R

$$W(R) \rightarrow R$$

\wp \curvearrowright Frobenius

$$\mathbb{F}_p[[z]], (z)$$

$$\mathbb{F}_p[[z]] \xrightarrow[z]{\text{algebra}} \mathbb{Z}_R \quad R \ni z_R$$

$$R[[z]] \longrightarrow R$$

$$\wp : r_0 + r_1 z + \dots \mapsto r_0^p + r_1^p z + \dots$$