23/4 LANCE GURNEY

What is a prism?

Let k be a field. Too many good cohomology theories for varieties X/k.

ℓ-adic, $\ell \neq 0 \in k$	$H^i_{\acute{e}t}(X, \mathbb{Z}_\ell) \in \mathrm{Rep}_{\mathbb{Z}_\ell}(\mathrm{Gal}(\bar{k}/k))$, $\mathrm{Gal}(\bar{k}/k) = G_k$
algebraic de Rham,	$H^i(X, \Omega^\bullet_{X/k}) \in \mathrm{Vect}^{\mathrm{Fil}}(k)$
crystalline $p = 0 \in k$,	$H^i_{crys}(X/W(k)) \in \mathrm{Mod}(W(k)[F])$
Betti, $k \hookrightarrow \mathbb{C}$,	$H^i_{sing}(X(\mathbb{C}), \mathbb{Z}) \in \mathrm{Ab}$

Relations

$$H^i_{sing}(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^i(X, \Omega^\bullet_{X/k}) \otimes_k \mathbb{C}$$

$$H^i_{sing}(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \simeq H^i_{\acute{e}t}(X, \mathbb{Z}_\ell)$$

Grothendieck

Motivic $\quad h^i(X) \in \mathrm{Mot}_k$

$$\begin{tikzcd} & \mathrm{Rep}_{\mathbb{Z}_\ell}(G_k) \\ & \mathrm{Vect}^{\mathrm{Fil}}(k) \\ \mathrm{Mot}_k \arrow[uur, "\text{ét}"] \arrow[ur, "\text{deRham}"] \arrow[r, "\text{crys}"] \arrow[dr, "\text{Betti}"'] & \mathrm{Mod}(W(k)[F]) \\ & \mathrm{Ab} \end{tikzcd}$$

NOT KNOWN TO EXIST

Existence would follow from resolution of standard conjectures.

Goal Find an explicit cohomology theory that determines "all" the others.

Prismatic cohomology determines all known p-adic cohomology theories for p-adic varieties.

$X / \mathbb{Z}_p$ smooth & proper

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p}) \ni H^i_{\text{ét}}(X_{\mathbb{Q}_p}, \mathbb{Z}_p)$$

$$\mathrm{Vect}^{\mathrm{Fil}}(\mathbb{Z}_p) \ni H^i(X, \Omega^\bullet_{X/\mathbb{Z}_p})$$

$$\mathrm{Mod}(\mathbb{Z}_p[F]) \ni H^i_{\mathrm{crys}}(X_{\mathbb{F}_p}/\mathbb{Z}_p)$$

Drinfeld, Bhatt-Scholze

Prismatization functor:

$$\left\{\begin{array}{c}\text{p-adic formal}\\ \text{schemes}\end{array}\right\} \longmapsto \left\{\text{p-adic stacks}\right\}$$

over $[\hat{\mathbb{A}}^1/\mathbb{G}_m, \mathrm{Spf}(\mathbb{Z}_p)]$

$$X \longmapsto X^{\Delta}$$

prism

prismatization

* $\Sigma = \mathrm{Spf}(\mathbb{Z}_p)^{\Delta}$
* $F : X^{\Delta} \to X^{\Delta}$ lift of Frobenius
* $\Delta \subseteq \Sigma$ a divisor

Prismatic cohomology:

$$f^{\Delta} : X^{\Delta} \to \Sigma$$

$$H^i_{\Delta}(X) = R^i f^{\Delta}_* \mathcal{O}_{X^{\Delta}} \in \mathrm{QCoh}^{\mathrm{isog}}(\mathcal{O}_\Sigma[F])$$

$$\mathrm{QCoh}^{\mathrm{isog}}(\mathcal{O}_\Sigma[F]) \xrightarrow{\text{et}} \mathrm{Rep}_{\mathbb{Z}_p}(G_K)$$

$$\mathrm{QCoh}^{\mathrm{isog}}(\mathcal{O}_\Sigma[F]) \xrightarrow{\text{dR}} \mathrm{Vect}^{\mathrm{Fil}}(\mathbb{Z}_p) \quad [+\varepsilon]$$

$$\mathrm{QCoh}^{\mathrm{isog}}(\mathcal{O}_\Sigma[F]) \xrightarrow{\text{crys}} \mathrm{Mod}(\mathbb{Z}_p[F])$$

Topics

1. Definitions / properties of prismatization & cohomology
2. de Rham / crystalline realization
3. Étale realization (Kisin F-crystals & crystalline reps)
4. q-de Rham cohomology

What is a prism?

Σ is the moduli space of prisms (i.e. universal prism)

Definition A prism is a pair $(A, \xi : I \to A)$

1. A is a δ-ring
2. $\xi : I \to A$ is a distinguished quasi-ideal

A δ-ring is a pair (A, δ_A), where $\delta_A : A \to A$

1. $\delta(ab) = a^p \delta(b) + b^p \delta(a) + p\delta(a)\delta(b)$
2. $\delta(a+b) = \delta(a) + \delta(b) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a^i b^{p-i}$
3. $\delta(1) = 0$

Point: $\varphi(a) = a^p + p\delta(a)$ is a ring homomorphism

$\varphi(a) = a^p \mod pA$ lifts Frobenius

1. $\mathbb{Z}$ $\quad \delta_{\mathbb{Z}}(a) = \dfrac{a - a^p}{p}$, $\quad \varphi(a) = a$

$$(\mathbb{Z}, \delta_{\mathbb{Z}}) \longrightarrow (A, \delta_A) \quad \text{for any } (A, \delta_A).$$

Follows from this that $A \ni p^i \neq 0 \;\; \forall i \geq 1$.

2. If A is p-torsion free, then

$$\varphi(a) \equiv a^p \mod pA$$

$$\Longleftrightarrow \quad \delta(a) = \frac{\varphi(a) - a^p}{p}$$

i.e. δ-ring $\Leftrightarrow$ lift of Frobenius

3. $\mathbb{Z}[t]$, $\varphi(t) = t^p$, $\delta(t) = 0$

4. $\mathbb{Z}[x]$, $\varphi(x) = x^p + px$, $\delta(x) = x$

5. $\mathbb{Z}[\zeta_n] \subseteq \mathbb{C}$ $\qquad (n, p) = 1$

Universal examples

J = arithmetic jet space

$$\begin{tikzcd} \delta\text{-rings} \arrow[r, "\text{forget } \delta"] & \text{Rings} \arrow[l, bend right, swap, "J"] \arrow[l, bend left, "W"] \end{tikzcd}$$

W = p-typical Witt vectors

$\boxed{1}$ $\quad J(\mathbb{Z}[x]) = \mathbb{Z}\{x\} = \mathbb{Z}[\underbrace{x, \delta(x), \delta^2(x), \ldots}_{\text{free vars}}]$

$$\varphi(\delta^i(x)) = (\delta^i(x))^p + p\delta^{i+1}(x)$$

$\boxed{2}$ J is a left adjoint, so if

$$R = \frac{\mathbb{Z}[x_i : i \in I]}{(f_j : j \in J)}$$

$$\Rightarrow J(R) = \frac{\mathbb{Z}[x_i, \delta^k(x_i) ; i \in I, k = 1, \dots]}{(f_j, \delta(f_j), \delta^2(f_j), \dots ; j \in J)}$$

$\boxed{3}$ $$W(R) = \mathrm{Hom}_{\mathrm{Rings}}(\mathbb{Z}\{x\}, R) = \prod_{i=0}^{\infty} R \quad \text{as ab. groups}$$

$$(r_0, r_1, \dots) + (s_0, s_1, \dots) = \left(r_0 + s_0, r_1 + s_1 - \sum \tfrac{1}{p}\binom{p}{i} r_0^i s_0^{p-i}, \dots\right)$$

$\boxed{4}$ $W(\mathbb{F}_p) = \mathbb{Z}_p$, $W(\mathbb{F}_{p^r}) = \mathbb{Z}_p[\zeta_{p^r-1}]$

$\boxed{5}$ $W(\mathbb{F}_p[t, t^{\frac{1}{p}}, t^{\frac{1}{p^2}}, \dots]) \simeq \mathbb{Z}_p[t, t^{\frac{1}{p}}, t^{\frac{1}{p^2}}, \dots]^{\wedge}_{p}$

Analogy

$\mathbb{Z}_p$, (p)	$\mathbb{F}_p[\vert z \vert]$, (z)
$\mathbb{Z}_p$-algebra R	$\mathbb{F}_p[\vert z \vert]$ algebra $R \ni z_R$, $z \longmapsto z_R$
$W(R) \longrightarrow R$	$R[\vert z \vert] \longrightarrow R$
φ Frobenius	$\varphi : r_0 + r_1 z + \dots \longmapsto r_0^p + r_1^p z + \dots$