NOTES ON PRISMS (DRAFT)

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Contents

Introduction

These notes will contain an overview of various things prismatic relative to a general complete discrete valuation ring with finite residue field.

The theory here can be found in the case of $\mathbf{Z}_p$ in Bhatt–Scholze [BS19], or Drinfeld [Dri21] and [Dri]. Of course, any errors are my own.

1. Witt vectors and δ-structures

For now and forever we fix a complete discrete valuation ring $\mathcal{O}$ with maximal ideal $\mathfrak{p}$, finite residue field $k = \mathcal{O}/\mathfrak{p}$ of cardinality q, a power of a prime p. When convenient we also fix a generator $\pi \in \mathcal{O}$ of $\mathfrak{p}$. We write $\mathrm{Alg}_{\mathcal{O}}$ for the category of $\mathcal{O}$-algebras and $\mathrm{Alg}_{\mathcal{O}}^{\mathfrak{p}}$ for the category of $\mathcal{O}$-algebras in which $\mathfrak{p}$ is nilpotent.

1.1. **Witt vectors and δ-structures.**

1.1.1. *Definition.* Let A be an $\mathcal{O}$-algebra. A δ-structure on A is a map $\delta_\pi : \mathrm{A} \to \mathrm{A}$ satisfying the following identities:

(1) $\delta_\pi(x+y) = \delta_\pi(x) + \delta_\pi(y) - \sum_{i=1}^{q-1} \frac{1}{\pi}\binom{q}{i} x^{q-i} y^i$. [1]
(2) $\delta_\pi(xy) = x^p \delta_\pi(y) + \delta_\pi(x) y^q + \pi \delta_\pi(x)\delta_\pi(y)$.
(3) $\delta_\pi(a) = (a - a^q)/\pi$ for $a \in \mathcal{O}$.

This does not depend on the uniformiser π chosen in the sense that if $\pi' = \lambda\pi$ for $\lambda \in \mathcal{O}^\times$ then a map δ_π satisfies the identities above if and only if the map $\delta_{\pi'} := \lambda^{-1}\delta_\pi$ satisfies the analogous identities with π replaced everywhere by $\lambda\pi = \pi'$. In any case, the purpose of this structure is realised when we define the map

$$\varphi : \mathrm{A} \to \mathrm{A} : x \mapsto \varphi(x) := x^q + \pi\delta_\pi(x)$$

which the reader readily checks is an $\mathcal{O}$-algebra homomorphism $\varphi : \mathrm{A} \to \mathrm{A}$ lifting the q-power Frobenius modulo $\mathfrak{p}$. A morphism of δ-rings is any $\mathcal{O}$-algebra homomorphism commuting with the δ maps and $\mathrm{Alg}_{\delta_{\mathcal{O}}}$ denotes the category of δ-rings.

1.1.2. *The torsion free case.* If A is $\mathfrak{p}$-torsion free then δ-structures on A are in bijective correspondence with $\mathcal{O}$-algebra homomorphisms $\varphi : \mathrm{A} \to \mathrm{A}$ lifting the q-power Frobenius via

$$\varphi \mapsto \delta : x \mapsto \frac{\varphi(x) - x^q}{\pi}.$$

If A is not $\mathfrak{p}$-torsion free then it really is extra structure but, as explained by Bhatt-Scholze, δ-structures are really 'derived' Frobenius lifts.

1.1.3. *Witt vectors and arithmetic jets.* The forgetful functor $\mathrm{Alg}_{\delta_{\mathcal{O}}} \to \mathrm{Alg}_{\mathcal{O}}$ admits both a left and a right adjoint. The right adjoint is given by the $\mathcal{O}$-Witt vectors $\mathrm{W}_{\mathcal{O}}$ and the left adjoint by the $\mathcal{O}$-arithmetic Jet ring $\mathrm{J}_{\mathcal{O}}$.

Composed with the forgetful functor these adjoints give a comonad and monad respectively on the category of $\mathcal{O}$-algebras and a coaction of $\mathrm{W}_{\mathcal{O}}$ (resp. action of $\mathrm{J}_{\mathcal{O}}$) on an $\mathcal{O}$-algebra is the same as a $\delta_{\mathcal{O}}$-structure. The coaction map of $\mathrm{W}_{\mathcal{O}}(\mathrm{R})$ on itself is denoted $w : \mathrm{W}_{\mathcal{O}}(\mathrm{R}) \to \mathrm{W}_{\mathcal{O}}(\mathrm{W}_{\mathcal{O}}(\mathrm{R}))$ and called the Artin-Hasse exponential.

[1] Here $\frac{1}{\pi}\binom{q}{i}$ for $1 \le i \le q-1$ denotes the unique element in $\mathcal{O}$ which multiplied by π gives $\binom{q}{i}$.

1.1.4. *Coordinates on the Witt vectors.* The Witt vector functor $\mathrm{W}_{\mathcal{O}}$ is co-represented by $\mathrm{J}_{\mathcal{O}}(\mathcal{O}[t]) = \mathcal{O}\{t\}$ which is, by definition, the free δ-ring on a single generator. As an $\mathcal{O}$-algebra, it is a polynomial algebra on countably many generators given by the elements $\delta_i := \delta^{\circ i}(t) \in \mathcal{O}\{t\}$:

$$\mathcal{O}\{t\} \xrightarrow{\sim} \mathcal{O}[\delta_0, \delta_1, \ldots].$$

These generators induce the 'Joyal coordinates' on the Witt vectors:

$$\mathrm{W}_{\mathcal{O}}(\mathrm{R}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}\{t\}, \mathrm{R}) \xrightarrow{\sim} \prod_{i=0}^{\infty} \mathrm{R} : f \mapsto (f(\delta_0), f(\delta_1), \ldots).$$

The $\mathcal{O}$-algebra structure on the infinite product $\prod_{i=0}^{\infty} \mathrm{R}$ induced by the isomorphism above is the unique functorial one such that:

(i) The projection:

$$\prod_{i=0}^{\infty} \mathrm{R} \to \mathrm{R} : (\delta_0, \delta_1, \delta_2 \ldots) \to \delta_0$$

is an $\mathcal{O}$-algebra homomorphism.

(ii) The map

$$\prod_{i=0}^{\infty} \mathrm{R} \to \prod_{i=0}^{\infty} \mathrm{R} : (\delta_0, \delta_1, \ldots) \mapsto (\delta_0^q + \pi\delta_1, \delta_1^q + \pi\delta_2, \ldots)$$

is an $\mathcal{O}$-algebra homomorphism. This homomorphism corresponds to the Frobenius φ on the Witt vectors.

There is a second set of coordinates on $\mathrm{W}_{\mathcal{O}}(\mathrm{R})$ called the 'Witt coordinates'. If we denote them by $d_i \in \mathcal{O}\{t\}$ for $i = 0, 1, \ldots$ then $d_0 = \delta_0 = t$ and the rest are defined inductively to be the unique elements of $\mathcal{O}\{t\}$ such that $\varphi^{\circ n}(t) \in \mathcal{O}\{t\}$ is given by the formula

$$\varphi^{\circ n}(t) = \sum_{i=0}^{n} \pi^i d_i^{q^{n-i}} = d_0^{q^n} + \pi d_1^{q^{n-1}} + \pi^2 d_2^{q^{n-2}} + \cdots + \pi^n d_n.$$

This induces a second isomorphism $\mathrm{W}_{\mathcal{O}}(\mathrm{R}) \xrightarrow{\sim} \prod_{i=0}^{\infty} \mathrm{R}$ and the resulting $\mathcal{O}$-algebra structure on $\prod_{i=0}^{\infty} \mathrm{R}$ is the unique functorial one such that the maps

$$g_n : \prod_{i=0}^{\infty} \mathrm{R} \to \mathrm{R} : (d_0, d_1, d_2 \ldots) \to \sum_{i=0}^{n} \pi^i d_i^{q^{n-i}}$$

are $\mathcal{O}$-algebra homomorphisms for $i \geq 0$. A coordinate free description of these maps

$$g_n : \mathrm{W}_{\mathcal{O}}(\mathrm{R}) \to \mathrm{R}$$

is iterates of the Frobenius φ^n composed with the canonical projection $\mathrm{W}_{\mathcal{O}}(\mathrm{R}) \to \mathrm{R}$ and are called the ghost maps.

1.1.5. *Teichmüller map.* The $\mathcal{O}$-algebra $\mathcal{O}[t]$ has a unique δ-structure with Frobenius lift $\varphi(t) = t^q$. By adjunction we find a unique δ-map

$$\mathcal{O}\{t\} \to \mathcal{O}[t]$$

and the induced map

$$[-] : \mathrm{R} \to \mathrm{W}_{\mathcal{O}}(\mathrm{R})$$

is the Teichmüller map. It is the unique multiplicative (but in general non-additive) section of the projection $W_{\mathcal{O}}(R) \to R$.

1.1.6. *Verschiebung.* The kernel of the projection $W_{\mathcal{O}}(R) \to R$ is denoted by $VW(R)$ and called the Verschiebung ideal. The restriction of the Frobenius to $VW(R)$ has image contained in $\mathfrak{p}W(R)$ and it can be lifted to a unique functorial isomorphism

$$\varphi : VW(R) \xrightarrow{\sim} \mathfrak{p} \otimes W(R).$$

The inverse of this isomorphism is called the Verschiebung map

$$V : \mathfrak{p} \otimes W(R) \xrightarrow{\sim} VW(R) \subset W(R).$$

We denote by V_π the map $V_\pi(w) = V(\pi \otimes w)$ which in terms of the Witt coordinates is given by

$$V_\pi(d_0, d_1, \ldots) = (0, d_0, d_1, \ldots).$$

The Verschiebung (so normalised) satisfies the relations:

(i) $\varphi(V_\pi(w)) = \pi w$,
(ii) $V_\pi(\varphi(w)w') = wV_\pi(w')$,
(iii) $V_\pi(w)V_\pi(w') = \pi V_\pi(ww')$.

1.1.7. *Finite length Witt vectors.* The image of the nth iterate V_π^n of the Verschiebung is denoted by $V^nW(R)$ and called the nth Verschiebung ideal. The quotient $W_{\mathcal{O},n}(R) := W_{\mathcal{O}}(R)/V^nW_{\mathcal{O}}(R)$ is the ring of length n Witt vectors and we have

$$W_{\mathcal{O},n}(R) \xrightarrow{\sim} \lim_n W_{\mathcal{O},n}(R).$$

This equips $W_{\mathcal{O}}(R)$ with a natural topology, or better, a natural pro-ring structure.

The maps δ, φ, V_π and w are compatible with this structure and induce maps

$$\delta, \varphi : W_{\mathcal{O},n+1}(R) \to W_{\mathcal{O},n}(R) \qquad V_\pi^i : W_{\mathcal{O},n}(R) \to W_{\mathcal{O},n+i}(R)$$

and

$$w : W_{\mathcal{O},n+m}(R) \to W_{\mathcal{O},n}(W_{\mathcal{O},m}(R))$$

and then short exact sequences

$$0 \to W_{\mathcal{O},n}(R) \xrightarrow{V_\pi^i} W_{\mathcal{O},n+i}(R) \to W_{\mathcal{O},i}(R) \to 0$$

for all $0 \leq i \leq n \leq \infty$ (where $i, n = \infty$ means the infinite length Witt vectors).

1.1.8. *Witt vectors as series.* It follows that using the Teichmüller and the Verschiebung we can uniquely write any Witt vector $w \in W_{\mathcal{O}}(R)$ as an infinite series

$$w = \sum_{i=0}^{\infty} V_\pi^i [r_i]$$

with $r_0, r_1, \ldots \in R$. The induced coordinates agree with the Witt coordinates defined earlier.

1.1.9. *Witt vectors, nilpotent ideals and étale maps.* If $\mathrm{R} \to \mathrm{R}'$ is an étale homomorphism then for all finite n and all homomorphisms $\mathrm{R} \to \mathrm{R}''$, the natural map

$$\mathrm{W}_{\mathcal{O},n}(\mathrm{R}') \otimes_{\mathrm{W}_{\mathcal{O},n}(\mathrm{R})} \mathrm{W}_{\mathcal{O},n}(\mathrm{R}'') \xrightarrow{\sim} \mathrm{W}_{\mathcal{O},n}(\mathrm{R}' \otimes_{\mathrm{R}} \mathrm{R}'')$$

is an isomorphism.

For $1 \leq n \leq \infty$ and $\mathrm{I} \subset \mathrm{R}$ an ideal we write $\mathrm{W}_{\mathcal{O},n}(\mathrm{I}) = \ker(\mathrm{W}_{\mathcal{O},n}(\mathrm{R}) \to \mathrm{W}_{\mathcal{O},n}(\mathrm{R}))$. Then:

(i) If I is nilpotent and $1 \leq n < \infty$ then $\mathrm{W}_{\mathcal{O},n}(\mathrm{I})$ is also nilpotent.
(ii) If $\mathfrak{p}$ is nilpotent in R then $\mathrm{V}_\pi^i \mathrm{W}_{\mathcal{O},n}(\mathrm{R}) \subset \mathrm{W}_{\mathcal{O},n+i}(\mathrm{R})$ is nilpotent, as is $\mathfrak{p}\mathrm{W}_{\mathcal{O},n}(\mathrm{R})$.
(iii) We have $\varphi(\mathrm{W}_{\mathcal{O},n}(\mathrm{I})) \subset \mathrm{W}_{\mathcal{O},n-1}(\mathrm{I}^q + \pi\mathrm{I})$ (use Joyal coordinates).

1.1.10. *Witt vectors of k-algebras.* If R is an $\mathcal{O}/\mathfrak{p} = k$-algebra then the Witt vector Frobenius φ coincides with $\mathrm{W}_{\mathcal{O}}(\mathrm{Fr}^q)$. This implies that the Verschiebung and Frobenius actually commute

$$\varphi(\mathrm{V}_\pi(w)) = \mathrm{V}_\pi(\varphi(w)) = \pi w.$$

In fact, this is equivalent to R being a k-algebra. In particular, if the Frobenius is an isomorphism on R, it is on $\mathrm{W}_{\mathcal{O}}(\mathrm{R})$ and $\mathrm{V}_\pi^i = \pi^i \varphi^{-i}$. In this case, every element of $\mathrm{W}_{\mathcal{O}}(\mathrm{R})$ can be written uniquely as a 'power series in π':

$$w = \sum_{i=0}^{\infty} [r_i]\pi^i.$$

Note that this implies that $\mathrm{W}_{\mathcal{O}}(\mathrm{R})$ is π-torsion free.

1.1.11. *The case $\mathcal{O} = \mathbf{F}_q[[\pi]]$.* If $\mathcal{O} = \mathbf{F}_q[[\pi]]$ is equi-characteristic and R is an $\mathbf{F}_q[[\pi]]$-algebra then the Teichmüller map

$$[-] : \mathrm{R} \to \mathrm{W}_{\mathbf{F}_q[[\pi]]}(\mathrm{R})$$

is an $\mathbf{F}_q$-linear ring homomorphism, that is it is not only multiplicative but also additive. It is not an $\mathbf{F}_q[[\pi]]$-algebra homomorphism, as $[\pi] \neq \pi$ in $\mathrm{W}_{\mathbf{F}_q[[\pi]]}(\mathrm{R})$. However, there is an induced $\mathbf{F}_q[[\pi]]$-linear map

$$\mathrm{R} \otimes_{\mathbf{F}_q} \mathbf{F}_q[[\pi]] \to \mathrm{W}_{\mathbf{F}_q[[\pi]]}(\mathrm{R}).$$

If R is π-adically complete (the case we are mainly interested in) then this map extends by continuity to a map

$$\mathrm{R}[[\pi]] \to \mathrm{W}_{\mathbf{F}_q[[\pi]]}(\mathrm{R})$$

which is in general is neither surjective nor injective. However, if R is perfect (i.e. the q-power Frobenius is an isomorphism) then it is an isomorphism. Moreover, for any R, $\mathrm{R}[[\pi]]$ has a unique δ-structure, given by $\delta(r) = 0$ for $r \in \mathrm{R} \subset \mathrm{R}[[\pi]]$, for which the $\mathrm{R}[[z]] \to \mathrm{W}_{\mathbf{F}_q[[\pi]]}(\mathrm{R})$ is a δ-homomorphism.

1.2. δ-structures on sheaves.

1.2.1. *Pro-rings and ind-affine sheaves.* Let $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{pro}}$ denote the category of pro-$\mathcal{O}$-algebras. We denote a general object of this category by

$$\text{“}\lim_{i\in \mathrm{I}} \mathrm{R}_i\text{”}.$$

Then the Yoneda embedding $\mathrm{R} \mapsto \mathrm{Spec}(\mathrm{R})$ extends to the category of pro-$\mathcal{O}$-algebras by

$$\text{“}\lim_{i\in \mathrm{I}} \mathrm{R}_i\text{”} \mapsto \operatorname*{colim}_{i\in \mathrm{I}} \mathrm{Spec}(\mathrm{R}_i)$$

and this functor is fully faithful. The essential image of this functor is the category of ind-affine sheaves $\mathrm{Aff}_{\mathcal{O}}^{\mathrm{ind}}$.

1.2.2. *Witt vectors of sheaves.* Given a sheaf X, writing it as its ‘Yoneda colimit’

$$\mathrm{X} \xrightarrow{\sim} \operatorname*{colim}_{\mathrm{Spec(R)}\to \mathrm{X}} \mathrm{Spec}(\mathrm{R})$$

we define

$$\mathrm{W}_{\mathcal{O},n}(\mathrm{X}) := \operatorname*{colim}_{\mathrm{Spec(R)}\to \mathrm{X}} \mathrm{Spec}(\mathrm{W}_{\mathcal{O},n}(\mathrm{R})) \quad \text{and} \quad \mathrm{W}_{\mathcal{O}}(\mathrm{X}) = \operatorname*{colim}_{n} \mathrm{W}_{\mathcal{O},n}(\mathrm{X}).$$

If $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$ is affine then

$$\mathrm{W}_{\mathcal{O}}(\mathrm{Spec}(\mathrm{R})) = \operatorname*{colim}_{n} \mathrm{Spec}(\mathrm{W}_{\mathcal{O},n}(\mathrm{R}))$$

is identified with the ind-affine scheme corresponding to the pro-ring “$\lim_{n} \mathrm{W}_{\mathcal{O},n}(\mathrm{R})$”.

1.2.3. *δ-structures on sheaves.* The functor $\mathrm{W}_{\mathcal{O}}$ on $\mathrm{Sh}_{\mathcal{O}}$ defines a monad (the variance has changed) and a δ-structure on sheaf is an action of this monad. We write $\mathrm{Sh}_{\delta_{\mathcal{O}}}$ for the category of δ-sheaves (that is sheaves equipped with a δ-structure) and note that the forgetful functor to the category of sheaves commutes limits, disjoint unions and filtered colimits. Moreover, $\mathrm{X} \mapsto \mathrm{W}_{\mathcal{O}}(\mathrm{X})$ is (by definition) left adjoint to the forgetful functor.

Of course, any δ-sheaf X has a lift of q-power Frobenius $\varphi : \mathrm{X} \to \mathrm{X}$.

1.2.4. *Arithmetic jet (pre)sheaves.* The forgetful functor from δ-sheaves to all sheaves wants to have a right adjoint given by the Jet space:

$$\mathrm{J}_{\mathcal{O}}(\mathrm{X}) := \lim_{n} \mathrm{X} \circ \mathrm{W}_{\mathcal{O},n}.$$

However $\mathrm{J}_{\mathcal{O}}(\mathrm{X})$ is not in general an fpqc sheaf as the functors $\mathrm{W}_{\mathcal{O},n}$ are not continuous for the fpqc topology (although they are continuous for the étale topology). Whenever the presheaf $\mathrm{J}_{\mathcal{O}}(\mathrm{X})$ defined above *is* a sheaf, the adjunction property holds – in particular for X a scheme (as in this case each presheaf $\mathrm{J}_{\mathcal{O},n}(\mathrm{X}) := \mathrm{X}\circ \mathrm{J}_{\mathcal{O},n}$ is itself a scheme). In general, we will see later that after ‘perfecting’ the functor $\mathrm{J}_{\mathcal{O}}$ becomes continuous.

1.2.5. *$\mathfrak{p}$-adic sheaves.* The terminal object $\mathrm{Spec}(\mathcal{O})$ in $\mathrm{Sh}_{\mathcal{O}}$ has a natural subsheaf

$$\mathrm{Spf}(\mathcal{O}) := \operatorname*{colim}_{i} \mathrm{Spec}(\mathcal{O}/\mathfrak{p}^i) \subset \mathrm{Spec}(\mathcal{O})$$

whose value on an $\mathcal{O}$-algebra R is the singleton if $\mathfrak{p}$ is nilpotent in R and empty otherwise. A sheaf $\mathrm{X} \in \mathrm{Sh}_{\mathcal{O}}$ is said to be $\mathfrak{p}$-adic if its structure map $\mathrm{X} \to \mathrm{Spec}(\mathcal{O})$ factors through $\mathrm{Spf}(\mathcal{O}) \subset \mathrm{Spec}(\mathcal{O})$. We will write $\mathrm{Sh}_{\mathcal{O}}^{\mathfrak{p}}$ for this category.

1.2.6. *$\mathfrak{p}$-adic δ-sheaves.* We now arrive at our final destination which is the category of $\mathfrak{p}$-adic δ-sheaves. Note that $\mathrm{Sh}_{\mathcal{O}}^{\mathfrak{p}}$ is stable under the functor $\mathrm{W}_{\mathcal{O}}$ as if $\mathfrak{p}$ is nilpotent in R it is also nilpotent in $\mathrm{W}_{n,\mathcal{O}}(\mathrm{R})$ so that $\mathrm{Spec}(\mathrm{W}_{\mathcal{O},n}(\mathrm{R}))$ is a $\mathfrak{p}$-adic sheaf and therefore so is $\mathrm{W}_{\mathcal{O}}(\mathrm{Spec}(\mathrm{R}))$. We denote by $\mathrm{Sh}_{\delta_{\mathcal{O}}}$ the category of $\mathfrak{p}$-adic sheaves equipped with a δ-structure and compatible morphisms. It will be useful later to note that if $\mathfrak{p}$ is nilpotent in R then the morphisms $\mathrm{W}_{\mathcal{O},n}(\mathrm{Spec}(\mathrm{R})) \to \mathrm{W}_{\mathcal{O}}(\mathrm{Spec}(\mathrm{R}))$ are representable by nilpotent immersions.

1.3. **Stacks and quasi-coherent modules.**

1.3.1. *'Algebraic' stacks.* A morphism of schemes $f : \mathrm{X} \to \mathrm{Y}$ is said to be fpqc if it is faithfully flat and a covering morphism for the fpqc topology.

A stack $\mathscr{X}$ is said to be algebraic if there exists a morphism $\mathrm{X} \to \mathscr{X}$ from a scheme X which is representable by fpqc morphisms. A morphism $f : \mathscr{X} \to \mathscr{Y}$ of stacks is said to be algebraic if for all affine schemes $\mathrm{Spec}(\mathrm{R}) \to \mathscr{Y}$ the stack $\mathscr{X} \times_{\mathscr{Y}} \mathrm{Spec}(\mathrm{R})$ is algebraic. Algebraic morphisms are preserved under composition, base change and satisfy fpqc descent.

1.3.2. *Quasi-coherent modules.* If X is a sheaf[2] ($\mathfrak{p}$-adic if you like, but for this it is not important) then a quasi-coherent module $\mathscr{M}$ on X is defined to be the following data: for all maps $f : \mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$, we are given an R-module

$$\mathscr{M}_f$$

and for all morphisms $h : \mathrm{Spec}(\mathrm{R}') \to \mathrm{Spec}(\mathrm{R})$ an isomorphism

$$h^*(\mathscr{M}_f) \xrightarrow{\sim} \mathscr{M}_{f\circ h}$$

satisfying the usual compatibility conditions, where the first h^* denotes the usual base change induced by $h : \mathrm{R} \to \mathrm{R}'$. We denote by QCoh(X) the category of quasi-coherent modules on X. We write $\mathscr{O}_\mathrm{X}$ for the quasi-coherent module $(\mathscr{O}_\mathrm{X})_f = \mathrm{R}$. Of course, $\mathrm{X} \mapsto \mathrm{QCoh}(\mathrm{X})$ is just the right Kan extension of its restriction to affine schemes.

When $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$, or more generally any scheme, QCoh(X) agrees with the usual notion. In general, if we can write $\mathrm{X} = \mathrm{colim}_i \, \mathrm{X}_i$ then

$$\mathrm{QCoh}(\mathrm{X}) \xrightarrow{\sim} \lim_i \mathrm{QCoh}(\mathrm{X}_i)$$

whence the observation that QCoh(X) may in general not be abelian. In particular, if $\mathrm{X} = \mathrm{Spf}(\mathcal{O})$ then QCoh(X) is equivalent to the category of $\mathfrak{p}$-adically complete $\mathcal{O}$-modules, which is not abelian. If we can write $\mathrm{X} = \mathrm{colim}_i \, \mathrm{X}_i$ in such a way that each $\mathrm{QCoh}(\mathrm{X}_i)$ is abelian and the transition maps induce exact functors on quasi-coherent modules e.g. for a scheme and an open cover by affines, but it works more generally.

If $f : \mathrm{X} \to \mathrm{Y}$ is a morphism of sheaves and $\mathscr{M}$ is a quasi-coherent module on Y then $f^*(\mathscr{M})$ is defined by setting, for any $h : \mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$

$$f^*(\mathscr{M})_h := \mathscr{M}_{f\circ h}.$$

[2]It could also be a presheaf - the definition never uses that X is a sheaf. The same definition also works for stacks or prestacks, with the usual modifications: if $\mathscr{X}$ is a (pre)stack then we ask for functor functors $\mathscr{X}(\mathrm{R}) \to \mathrm{Mod}(\mathrm{R})$ satisfying various compatibilities (rather than just maps $\mathrm{X}(\mathrm{R}) \to \mathrm{Mod}(\mathrm{R})$ for presheaves X).

We say that $\mathscr{M}$ is a vector bundle if $f^*(\mathscr{M})$ is a finite projective R-module for all $f : \mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$ and that $\mathscr{M}$ and a line bundle if it is finite projective of rank one.

An important case is when $\mathrm{X} = \mathrm{colim}_i \, \mathrm{Spec}(\mathrm{R}_i)$ is an ind-affine scheme. Then a line bundle $\mathscr{L}$ on X is by definition a compatible collection L_i of rank one projective R_i-modules. If moreover the transition maps $\mathrm{R}_i \to \mathrm{R}_j$ have kernels generated by nilpotent elements, we see that as soon as one L_i is free all of them are and $\mathscr{L} \xrightarrow{\sim} \mathscr{O}_\mathrm{X}$. In particular, any affine open cover of *one* $\mathrm{Spec}(\mathrm{R}_i)$ induces unique compatible open covers of *all* the $\mathrm{Spec}(\mathrm{R}_j)$. Thus in this case, there is a relative affine open cover of X with the property that the pull-back of $\mathscr{L}$ to this open cover is free.

Given a quasi-coherent module $\mathscr{M}$ on X we can turn it into a 'physical' object over X, that is an actual sheaf over X, as follows. We define $\mathbf{V}(\mathscr{M})$ to be the sheaf over X whose set of sections over a sheaf $f : \mathrm{Y} \to \mathrm{X}$ is

$$\mathbf{V}(\mathscr{M})(\mathrm{Y}) := \mathrm{Hom}_{\mathrm{QCoh}(\mathrm{Y})}(\mathscr{O}_\mathrm{Y}, f^*(\mathscr{M})).$$

In this way, we see that $\mathbf{V}(\mathscr{O}_\mathrm{X}) \xrightarrow{\sim} \mathbf{A}^1_\mathrm{X}$. That is

$$\mathbf{V}(\mathscr{O}_\mathrm{X})(\mathrm{Y}) = \mathrm{Hom}_{\mathrm{QCoh}(\mathrm{Y})}(\mathscr{O}_\mathrm{Y}, f^*(\mathscr{O}_\mathrm{X})) = \mathrm{Hom}_\mathrm{X}(\mathrm{Y}, \mathbf{A}^1_\mathrm{X}).$$

More generally, a quasi-coherent module $\mathscr{E}$ on X is a vector bundle if and only if $\mathbf{V}(\mathscr{E})$ is locally[3] isomorphic to $\mathbf{V}(\mathscr{O}^n_\mathrm{X}) = \mathbf{A}^n_\mathrm{X}$.

2. Prisms

2.1. Distinguished elements and quasi-ideals.

2.1.1. *Distinguished elements.* Let R be an $\mathcal{O}$ in which $\mathfrak{p}$ is nilpotent, i.e. $\mathrm{R} \in \mathrm{Alg}^{\mathfrak{p}}_{\mathcal{O}}$, and consider the Witt vectors W(R). We say than an element $\xi \in \mathrm{W}(\mathrm{R})$ is distinguished if one of the following equivalent conditions hold:

- (i) $\xi = [\xi_0] + \mathrm{V}_\pi[\xi_1] + \cdots$ with $\xi_0 \in \mathrm{R}$ nilpotent and $\xi_1 \in \mathrm{R}$ invertible,
- (ii) $\xi = [\xi_0] + \mathrm{V}_\pi(w)$ with $\xi_0 \in \mathrm{R}$ nilpotent and $w \in \mathrm{W}(\mathrm{R})$ a unit,
- (iii) The image of ξ in $\mathrm{W}_n(\mathrm{R})$ is nilpotent for each $n \geq 1$ and $\delta(\xi) \in \mathrm{W}(\mathrm{R})$ is a unit.

The fundamental example of a distinguished element is $\pi \in \mathrm{W}(\mathrm{R})$.

2.1.2. **Proposition** (Properties of distinguished elements)**.** *Let* R *be an* $\mathcal{O}/\mathfrak{p}^r$*-algebra and let* $\xi \in \mathrm{W}(\mathrm{R})$.

- (i) ξ *is distinguished if and only if* $\varphi(\xi)$ *is distinguished.*
- (ii) *If* ξ *is distinguished and* $u \in \mathrm{W}(\mathrm{R})$ *then* $\xi' := u\xi$ *is distinguished if and only if* u *is a unit.*
- (iii) *If* $\xi^{q^n} = 0 \bmod \mathrm{VW}(\mathrm{R})$ *for some* $n \geq 0$, *then* $\varphi^{n+1}(\xi) = u\pi$ *where* u *is a unit.*
- (iv) *If* ξ *is distinguished,* $\xi^{q^n} = 0 \bmod \mathrm{VW}(\mathrm{R})$ *for some* $n \geq 0$ *and* $u\xi = \xi$ *then* $\varphi^{n+r+1}(u) = 1$.
- (v) *If* $f : \mathrm{R} \to \mathrm{R}'$ *is a homomorphism and* $\xi \in \mathrm{W}(\mathrm{R})$ *is distinguished then* $\mathrm{W}(f)(\xi) \in \mathrm{W}(\mathrm{R}')$ *is distinguished.*

[3]Here locally is meant in the most general sense: after base change along an epimorphism $\mathrm{X}' \to \mathrm{X}$ of fpqc sheaves.

Proof. (i) As $\varphi(\xi) = \xi^q + \pi\delta(\xi)$ and π is topologically nilpotent, we see that ξ is topologically nilpotent if and only if $\varphi(\xi)$ is topologically nilpotent. Similarly,

$$\delta(\varphi(\xi)) = \varphi(\delta(\xi)) = \delta(\xi)^q + \pi(\delta(\delta(\xi)))$$

so that $\delta(\varphi(\xi))$ is a unit if and only if $\delta(\xi)$ is a unit.

(ii) If ξ is distinguished and $u \in \mathrm{W(R)}$ is a unit then $\xi' = u\xi$ is topologically nilpotent and

$$\delta(u\xi) = u^q\delta(\xi) + \varphi(\xi)\delta(u)$$

is a unit as $u^q\delta(\xi)$ is a unit and $\varphi(\xi)$ is topologically nilpotent.

Conversely, if $u \in \mathrm{W(R)}$ is any element such that $\xi' = u\xi$ is distinguished then the same equality

$$\delta(u\xi) = u^q\delta(\xi) + \varphi(\xi)\delta(u)$$

shows that u^q and hence u must be a unit.

(iii) We have $\xi = [\xi_0] + \mathrm{V}_\pi(w)$ where w is a unit and $\xi^{q^n} = 0 \bmod \mathrm{VW(R)}$ is equivalent to $\xi_0^{q^n} = 0$ in R. Therefore,

$$\varphi^{n+1}(\xi) = \varphi^{n+1}([\xi_0]) + \varphi^{n+1}(\mathrm{V}_\pi(w)) = [\xi_0^{q^n+1}] + \pi\varphi^n(w) = \pi\varphi^n(w).$$

(iv) By (iii) we are reduced to showing that if $u\pi = \pi$ then $\varphi^r(u) = 1$. If $r = 1$, so that $\pi = 0$ in R, then $\varphi \circ \mathrm{V}_\pi = \mathrm{V}_\pi \circ \varphi = \pi$ so that

$$0 = p(u-1) = \varphi(\mathrm{V}_\pi(u-1)) = \mathrm{V}_\pi(\varphi(u-1)).$$

As V_π is injective this implies that $\varphi(u) = 1$.

In general, this shows that $\varphi(u-1) \in \mathrm{W}(\mathfrak{p}\mathrm{R}) = \ker(\mathrm{W(R)} \to \mathrm{W}(\mathrm{R}/\mathfrak{p}))$. However, we have $\varphi(\mathrm{W}(\mathfrak{p}^i\mathrm{R})) \subset \mathrm{W}(\mathfrak{p}^{i+1}\mathrm{R})$ so that if $\mathfrak{p}^r = 0$ we get:

$$\varphi^r(u-1) = \varphi^{r-1}(\varphi(u-1)) \in \varphi^{r-1}(\mathrm{W}(\mathfrak{p}\mathrm{R})) \subset \mathrm{W}(\mathfrak{p}^r\mathrm{R}) = 0.$$

(v) Clear. □

2.1.3. *Distinguished quasi-ideals over Witt vectors.* Write $\mathrm{S} = \mathrm{Spec(R)}$. A distinguished quasi-ideal on W(S) is a line bundle $\mathscr{I}$ on W(S) equipped with a map

$$\xi : \mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}}$$

such that on one (or any) open cover[4] over which $\mathscr{I} \xrightarrow{\sim} \mathscr{O}_{\mathrm{W(S)}}$ is trivialised,

$$\xi \in \mathrm{Hom}(\mathscr{O}_{\mathrm{W(S)}}, \mathscr{O}_{\mathrm{W(S)}}) = \mathrm{W(R)}$$

is distinguished. A morphism of distinguished quasi-ideals

$$u : (\xi : \mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}}) \to (\xi' : \mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}})$$

is any morphism $u : \mathscr{I} \to \mathscr{I}'$ such that $\xi' \circ u = \xi$.

If $\mathrm{S}' = \mathrm{Spec(R')}$ and $f : \mathrm{S}' \to \mathrm{S}$ is a morphism then we write

$$f^*(\xi : \mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}}) = (f^*(\xi) : f^*(\mathscr{I}) \to \mathscr{O}_{\mathrm{W(S)}})$$

which is again a distinguished quasi-ideal by (v) of (2.1.2).

Given a distinguished quasi-ideal $(\xi : \mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}})$ and writing $i : \mathrm{S} \to \mathrm{W(S)}$ for the natural closed immersion let us set $i^*(\mathscr{I}) = \mathscr{I}_0$ and $\xi_0 = i^*(\xi)$. Then we

[4] Such open covers exist because the transition maps $\mathrm{Spec}(\mathrm{W}_n(\mathrm{R})) \to \mathrm{Spec}(\mathrm{W}_{n+1}(\mathrm{R}))$ are nilpotent immersions. Any open subscheme of any Spec(R) lifts uniquely to a compatible family of open subschemes of the $\mathrm{Spec}(\mathrm{W}_n(\mathrm{R}))$ and hence to a (representable) open subscheme of the colimit W(Spec(R)). Indeed, if $\mathrm{U} \subset \mathrm{Spec(R)}$ is an open immersion then $\mathrm{W(U)} \to \mathrm{W(Spec(R))}$ is also an open immersion. Similarly, a line bundle on W(Spec(R)) is trivial if and only if its restriction to Spec(R) is trivial as trivialisations can always be lifted along nilpotent immersions.

have a morphism of line bundles $(\xi_0 : \mathscr{I}_0 \to \mathscr{O}_{\mathrm{S}})$ over S. As $(\xi : \mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}})$ is distinguished it follows that $\xi_0^{\otimes q^n}$ is the zero map for some $n \geq 0$.

2.1.4. *Remark.* A distinguished quasi-ideal on W(S) for S = Spec(R) with $\mathfrak{p}$ nilpotent in R is the same as a projective rank one W(R)-module I, together with a map $\xi : \mathrm{I} \to \mathrm{W(R)}$ such that, Zariski locally on R, ξ sends a generator of I to a distinguished element of W(R).

2.1.5. *Principal distinguished quasi-ideals.* If $\xi \in \mathrm{W(R)} = \mathrm{Hom}(\mathscr{O}_{\mathrm{W(S)}}, \mathscr{O}_{\mathrm{W(S)}})$ is a distinguished element then $(\xi : \mathscr{O}_{\mathrm{W(S)}} \to \mathscr{O}_{\mathrm{W(S)}})$ is a distinguished quasi-ideal and a distinguished quasi-ideal is said to be principal and generated by $\xi \in \mathrm{W(R)}$ if it is isomorphic to one of this form.

2.1.6. **Proposition** (Properties of distinguished quasi-ideals over Witt vectors)**.** *Let* $\mathrm{S} = \mathrm{Spec(R)}$ *where* $\mathfrak{p}^r = 0$ *in* R.

(i) *Every morphism of distinguished quasi-ideals over* W(S) *is an isomorphism.*
(ii) *If* $(\xi : \mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}})$ *is a distinguished quasi-ideal then so is* $(\varphi^*(\xi) : \varphi^*(\mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}})$*, and moreover it is principal.*
(iii) *If* $(\xi : \mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}})$ *is a distinguished quasi-ideal and* $\xi_0^{\otimes q^n} = 0$ *for some* $n \geq 0$ *then there exists an isomorphism*

$$(i_{\mathfrak{p}} : \mathfrak{p} \otimes \mathscr{O}_{\mathrm{W(S)}} \to \mathscr{O}_{\mathrm{W(S)}}) \xrightarrow{\sim} \varphi^{n+1*}(\xi : \mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}}).$$

(iv) *If* u *is an automorphism of a distingiushed quasi-ideal* $(\xi : \mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}})$ *with the property that* $\xi_0^{\otimes q^n} = 0$ *for some* $n \geq 1$*, then* $\varphi^{n+r+1*}(u) = \mathrm{id}$.

Proof. (i) It suffices to work locally and so we may assume that $\mathscr{I}$ is principal in which case it follows (ii) of (2.1.2).

(ii) The line bundle $\mathscr{I}$ and the morphism $\xi : \mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}}$ correspond to a compatible collection of rank one projective $\mathrm{W}_n(\mathrm{R})$-modules I_n and compatible maps $\xi_n : \mathrm{I}_n \to \mathrm{W}_n(\mathrm{R})$ and the claim is that the compatible collection $\varphi^*(\mathrm{I}_{n+1})$ of $\mathrm{W}_n(\mathrm{R})$-modules is free.

First, we define for each $n \geq 1$ a map

$$\mathrm{I}_{n+1} \to \mathrm{W}_n(\mathrm{R})/\xi_n(\mathrm{I}_n) : i \mapsto \delta_\pi(\xi_{n+1}(i)) \bmod \xi_n(\mathrm{I}_n).$$

We claim that this map is additive and φ-linear. In proving this, for clarity we will denote the reduction maps $\mathrm{W}_{n+1}(\mathrm{R}) \to \mathrm{W}_n(\mathrm{R})$ and $\mathrm{I}_{n+1} \to \mathrm{I}_n$ by $x \mapsto \overline{x}$. By definition we have: $\overline{\xi_{n+1}(i)} = \xi_n(\bar{i})$.

Now, for additivity we have:

$$\begin{aligned}
\delta_\pi(\xi_{n+1}(i+j)) &= \delta_\pi(\xi_{n+1}(i) + \xi_{n+1}(j)) \\
&= \delta_\pi(\xi_{n+1}(i)) + \delta_\pi(\xi_{n+1}(j)) + \sum_{k=1}^{q-1} \frac{1}{\pi}\binom{q}{k} \overline{\xi_{n+1}(i)^{q-k}\xi_{n+1}(j)^k} \\
&= \delta_\pi(\xi_{n+1}(i)) + \delta_\pi(\xi_{n+1}(j)) + \sum_{k=1}^{q-1} \frac{1}{\pi}\binom{q}{k} \xi_n(\bar{i})^{q-k}\xi_n(\bar{j})^k \\
&= \delta_\pi(\xi_{n+1}(i)) + \delta_\pi(\xi_{n+1}(j)) \bmod \xi_n(\mathrm{I}_n)
\end{aligned}$$

For the semi-linearity we have:

$$\begin{aligned}\delta_\pi(\xi_{n+1}(wi)) &= \delta_\pi(w\xi_{n+1}(i))\\ &= \varphi(w)\delta_\pi(\xi_{n+1}(i)) + \delta_\pi(w)\overline{\xi_{n+1}(i)}^q\\ &= \varphi(w)\delta_\pi(\xi_{n+1}(i)) + \delta_\pi(w)\xi_n(\bar{i})^q\\ &= \varphi(w)\delta_\pi(\xi_{n+1}(i)) \bmod \xi_n(\mathrm{I}_n)\end{aligned}$$

Therefore, we have defined a φ-linear map $\mathrm{I}_{n+1} \to \mathfrak{p} \otimes \mathrm{W}_n(\mathrm{R})/\xi_n(\mathrm{I}_n)$ which induces a linear map

$$\varphi^*(\mathrm{I}_{n+1}) \otimes_{\mathrm{W}_n(\mathrm{R})} \mathrm{W}_n(\mathrm{R})/(\xi_n(\mathrm{I}_n)) \to \mathfrak{p} \otimes \mathrm{W}_n(\mathrm{R})/\xi_n(\mathrm{I}_n).$$

This is a homomorphism of rank one projective $\mathrm{W}_n(\mathrm{R})$-modules and it is an isomorphism as it comes from a distinguished quasi-ideal: locally I_{n+1} admits a generator i with the property that $\delta_\pi(\xi_{n+1}(i))$ is invertible in $\mathrm{W}_n(\mathrm{R})$.

Therefore, $\varphi^*(\mathrm{I}_{n+1})$ is free and as $\mathrm{W}_{n+1}(\mathrm{R}) \to \mathrm{W}_n(\mathrm{R})$ is surjective with nilpotent kernel, it follows that we can compatibly lift generators and hence $\varphi^*(\mathscr{I})$ is free.

(iii) By induction we are reduced to the case $n = 0$. As in (ii) let $\xi : \mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}}$ correspond to the compatible system $\xi_m : \mathrm{I}_m \to \mathrm{W}_m(\mathrm{R})$. As $\xi_0 = 0$ it follows that ξ_m factors through $\mathrm{V}_\pi\mathrm{W}_{m-1}(\mathrm{R}) \subset \mathrm{W}_m(\mathrm{R})$ and so we can write each ξ_m uniquely as $\mathrm{V}_\pi \circ \beta_m$ where $\beta_m : \mathrm{I}_m \to \mathrm{W}_{m-1}(\mathrm{R})$ is a φ-linear map and by uniqueness we have $\overline{\beta_m} = \beta_{m-1}$.

We then see that $\varphi^*(\xi_m) : \varphi^*(\mathrm{I}_m) \to \mathrm{W}_{m-1}(\mathrm{R})$ is the linearisation of the φ-linear map $\pi\beta_m : \mathrm{I}_m \to \mathrm{W}_{m-1}(\mathrm{R})$. The linearisations of the β_m now define a compatible collection of maps

$$\varphi^*(\mathrm{I}_m) \overset{\pi\otimes\beta_m}{\to} \mathfrak{p} \otimes \mathrm{W}_{m-1}(\mathrm{R})$$

such that $\varphi^*(\xi_m) = i_\mathfrak{p} \circ (\pi \otimes \beta_m)$. Therefore, we have a morphism of distinguished quasi-ideals

$$(\varphi^*(\xi) : \mathscr{I} \to \mathscr{O}_{\mathrm{W(S)}}) \to (i_\mathfrak{p} : \mathfrak{p} \otimes \mathscr{O}_{\mathrm{W(S)}} \to \mathscr{O}_{\mathrm{W(S)}})$$

which by (i) is an isomorphism.

(iv) Arguing locally this follows from (iv) of (2.1.2). □

2.2. **Prisms and Σ.**

2.2.1. *Distinguished quasi-ideals over general sheaves.* Let X be a $\mathfrak{p}$-adic δ-sheaf. A distingiushed quasi-ideal over X is a morphism of line bundles $(\xi : \mathscr{I} \to \mathscr{O}_\mathrm{X})$ such that, for all rings R and all morphisms Spec(R) → X, the pull-back of $(\xi : \mathscr{I} \to \mathscr{O}_\mathrm{X})$ along the δ-map W(Spec(R)) → X induced by adjunction is a distinguished quasi-ideal in the sense of (2.1.3).

2.2.2. *Prisms.* A prism is a pair $(\mathrm{X}, (\xi : \mathscr{I} \to \mathrm{X}))$ where X is a $\mathfrak{p}$-adic δ-sheaf and $(\xi : \mathscr{I} \to \mathrm{X})$ is a distinguished quasi-ideal. A morphism of prisms

$$(f, u) : (\mathrm{X}, (\xi_\mathrm{X} : \mathscr{I}_\mathrm{X} \to \mathrm{X})) \to (\mathrm{Y}, (\xi_\mathrm{Y} : \mathscr{I} \to \mathrm{Y}))$$

consists a δ-morphism $f : \mathrm{X} \to \mathrm{Y}$ and a morphism of distinguished quasi-ideals

$$u : f^*(\mathscr{I}_\mathrm{Y}) \to \mathscr{I}_\mathrm{X}.$$

Note that u is necessarily an isomorphism.

2.2.3. *Drinfeld's stack* Σ. The R-points of the $\mathfrak{p}$-adic stack Σ are given by

$$\Sigma(\mathrm{R}) := \{\text{groupoid of distinguished quasi-ideals } (\xi : \mathscr{I} \to \mathscr{O}_{\mathrm{W}(\mathrm{Spec}(\mathrm{R}))})\}.$$

We denote by $\varphi_\Sigma : \Sigma \to \Sigma$ the morphism which on R-points is induced by pull-back along the Frobenius $\mathrm{W}(\mathrm{Spec}(\mathrm{R})) \to \mathrm{W}(\mathrm{Spec}(\mathrm{R}))$.

2.2.4. Σ *in terms of distinguished elements.* Write $\mathbf{W}$ for the affine group scheme whose R-points is the ring of Witt vectors $\mathbf{W}(\mathrm{R}) := \mathrm{W}(\mathrm{R})$. Note that $\mathbf{W}$ is isomorphic to the affine scheme $\mathrm{Spec}(\mathcal{O}\{t\})$.

Abusing notation we will also write $\mathbf{W}$ for $\mathbf{W} \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spf}(\mathcal{O})$.

Now, let $\mathbf{W}_{\mathrm{dist}} \subset \mathbf{W}$ denote the subsheaf whose R-points are those $\xi \in \mathbf{W}(\mathrm{R}) = \mathrm{W}(\mathrm{R})$ which are distinguished. Thus, we have

$$\mathbf{W}_{\mathrm{dist}} = \operatorname*{colim}_{n,m} \mathrm{Spec}(\mathcal{O}[t, \delta^{\circ 1}(t)^{\pm 1}, \delta^{\circ 2}(t), \ldots]/(\mathfrak{p}^n, t^m)).$$

We write $\mathbf{W}^\times \subset \mathbf{W}$ for the subsheaf whose set of R-points is the group of units in $\mathbf{W}(\mathrm{R}) = \mathrm{W}(\mathrm{R})$ (Note, can be shown that $\mathrm{W}^\times \to \mathrm{W}$ is an inverse limit of open immersions).

Then, $\mathbf{W}^\times$ acts on $\mathbf{W}_{\mathrm{dist}}$ via multiplication and the map

$$\mathbf{W}_{\mathrm{dist}} \to \Sigma$$

sending a distinguished element to the associated principal distinguished quasi-ideal identifies Σ with the stack-theoretic quotient

$$\mathbf{W}_{\mathrm{dist}}/\mathbf{W}^\times \xrightarrow{\sim} \Sigma.$$

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