Electoral Equilibria under Scoring Voting Rules: Beyond Plurality

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Introduction

Political candidates running in an election must decide where they stand on the ideological spectrum.

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- If so, will the candidates cluster together, advocating identical or similar policy positions, or will they diverge in order to appeal to different groups of voters?

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- Do equilibrium situations exist in the candidate positioning game?
- If so, will the candidates cluster together, advocating identical or similar policy positions, or will they diverge in order to appeal to different groups of voters?
- In particular, how does this depend on the voting system in use?
  - Plurality – voters vote once for their favourite candidate
  - Approval – voters either “approve" or “disapprove" of each candidate
  - Antiplurality – voters vote against their least favourite candidate

The model

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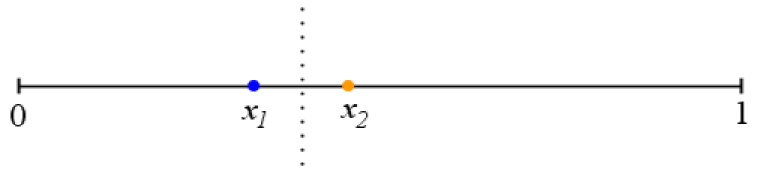
- There are m candidates. A *profile* is an m-vector $x = (x_1, \ldots, x_m) \in [0,1]^m$ specifying each candidate's position: x_i is candidate i's position.
- Voters rank all candidates by their ideological distance.



- Candidates aim to maximise their share of the vote (Downs, 1957).

Nash equilibrium

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Two kinds of Nash equilibria are of interest:

- A *convergent* Nash equilibrium (CNE) occurs when all candidates adopt the same ideological position.
- A *non-convergent* Nash equilibrium (NCNE) is when not all candidate positions are the same.

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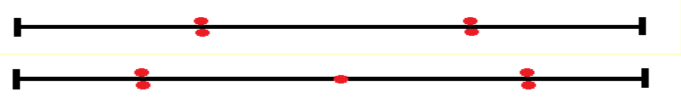
- For $m = 2$ we have CNE (Hotelling, 1929) — known as *the median voter theorem*.

- For $m = 3$ no Nash equlibria exist;
- For $m = 4, 5$ there is a unique NCNE;



- For $m \geq 6$ there are infinitely many NCNE (Eaton and Lipsey, 1975).

Scoring rules

- We consider a more general class of voting systems called *positional scoring rules*.
- Each voter ranks all the candidates.
- The candidate ranked i-th receives s_i points, according to an m-vector $s = (s_1, s_2, \ldots, s_m)$ of real numbers.

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- Require that $s_1 \geq \cdots \geq s_m$ and $s_1 > s_m = 0$, i.e., the scores are decreasing and first is better than last. E.g.,
  - Plurality: $s = (1, 0, 0, \ldots, 0)$
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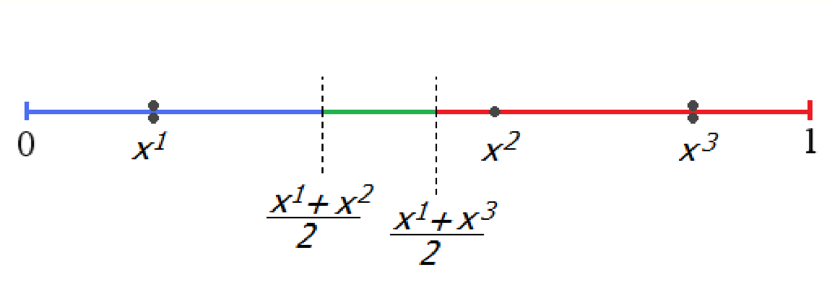
  - Antiplurality: $s = (1, 1, 1, \dots, 1, 0)$
- The candidates' overall scores are then calculated by integrating across all voters.
- Candidates are *score (share) maximisers*.

Scoring rules with ties

- If two or more candidates occupy the same policy position the voters will be indifferent between them.
- Voters break ties by a fair lottery. E.g., flipping a coin to decide the order of two identical candidates.

Workings of a scoring rule

Five-candidate election. Scoring rule $s = (s_1, s_2, s_3, s_4, s_5)$.



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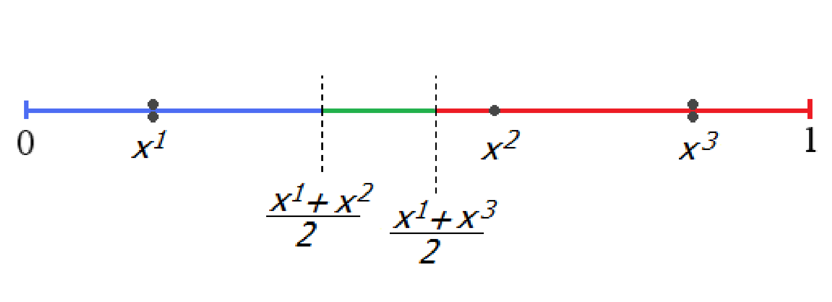

The score of a candidate positioned at x^1 would be

$$\frac{s_1+s_2}{2}\frac{x_1+x_2}{2}+\frac{s_2+s_3}{2}\frac{x_3-x_2}{2}+\frac{s_4+s_5}{2}\left(1-\frac{x_1+x_3}{2}\right).$$

Convergent Nash equilibria

Theorem (Cox, 1987). For m candidates and scoring rule s, a profile $x = (x^*, \ldots, x^*)$ is a CNE if and only if

$$c(s,m) \leq x^* \leq 1 - c(s,m), \qquad (1)$$

where $c(s,m) = \dfrac{s_1 - \bar{s}}{s_1 - s_m}$ is the *c-value* (with $\bar{s} = \frac{1}{m}\sum_{i=1}^{m} s_i$ being the average score).

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- $c(s,m)$ is number between 0 and 1. It is a measure of the competing incentives.
- If $c(s,m) > 1/2$ (best-rewarding rule[1]), the inequality (1) cannot hold. So no CNE exist.
- If $c(s,m) \leq 1/2$ (worst-punishing or intermediate rule), any $x^* \in [c(s,m), 1 - c(s,m)]$ is a CNE.

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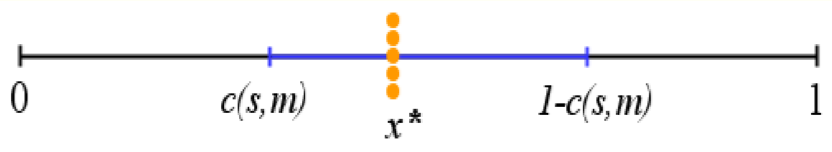

- The blue interval is the set of valid equilibrium positions.
- The size of the blue interval depends on the rule. The more best-rewarding the rule, the smaller the interval.
- If $c(s, m) > 1/2$ then there is no blue interval – no CNE exist.

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  - If $c(s, 3) > 1/2$ no equilibria exist.
- What about the case of four candidates? This is the first challenge.

The four-candidate case

Theorem (C.-S. 2011). In a four-candidate election under scoring rule $s = (s_1, s_2, s_3, s_4)$, NCNE exist iff both the following conditions are satisfied:

- a) $s_1 > s_2 = s_3$;
- b) $c(s, 4) > 1/2$.

Moreover, the NCNE is unique and symmetric with two candidates at

$$x_1 = \frac{s_1}{4(s_1 - s_2)}$$

and two at

$$x_2 = 1 - x_1.$$

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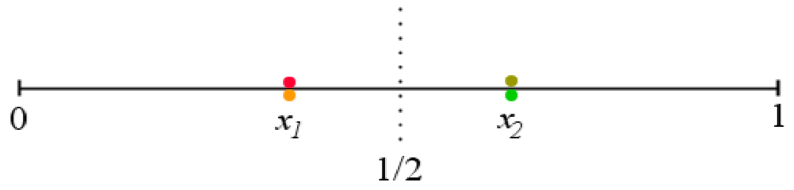

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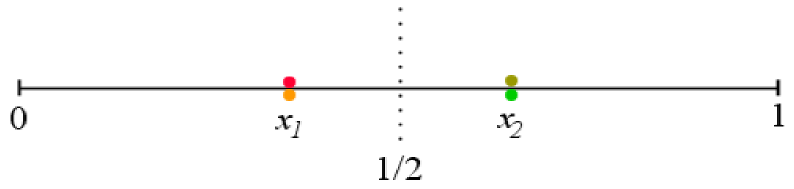

- The distance between the candidates depends on the rule. As $c(s,4) \to 1/2$, the positions converge to the half-way point.
- When $c(s,4)$ falls below 1/2, only CNE exist, given by the previous theorem.
- If $c(s,4) > 1/2$ but $s_2 \neq s_3$ then no equilibria of either kind exist.

A similar result exists for $m = 5$. The extra candidate is at $1/2$.

The general case

What about the general case of m candidates?

- For $m \geq 6$, a complete characterisation of scoring rules admitting NCNE is not known.
- What general properties of NCNE can we find?
- What can we say about particular classes of scoring rules?

Dispersion of NCNE

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- The higher $c(s,m)$, the more dispersion in a NCNE.
- E.g., the c-value is maximal under plurality, where $c(s,m) = 1 - 1/m$. The theorem then gives $q \geq \lceil m/2 \rceil$.

Repeated highest scores

Theorem (C.-S. 2011). Given a scoring rule s, let $1 \leq k \leq m - 1$ be such that $s_1 = \cdots = s_k > s_{k+1}$. Then a necessary condition for NCNE is that each of the most extreme occupied positions to each side has more than k candidates.

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Example. Consider k-approval, given by

$$s = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0).$$

If $k \geq \lfloor m/2 \rfloor$ then there are no NCNE.

Concave scores

We say that the score vector $s = (s_1, \ldots, s_m)$ is concave if

$$s_1 - s_2 \geq s_2 - s_3 \geq \cdots \geq s_{m-1} - s_m.$$

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- Exceptions: rules where the initial nonconstant part of s is Borda and the constant part is longer than the nonconstant part, e.g. plurality rule (it has NCNE) or $s = (2, 1, 0, 0, 0, 0)$.
- Borda itself, however, is not an exception. Hence, **Borda has no NCNE**.

Symmetric scores

We say a rule $s = (s_1, \ldots, s_m)$ is symmetric if

$$s_i - s_{i+1} = s_{m-i} - s_{m-i+1}$$

for $1 \leq i \leq \lfloor m/2 \rfloor$

- Compare the differences between adjacent components: $s = (s_1, \ldots, s_i, s_{i+1}, \ldots, , s_{m-i}, s_{m-i+1}, \ldots, s_m)$
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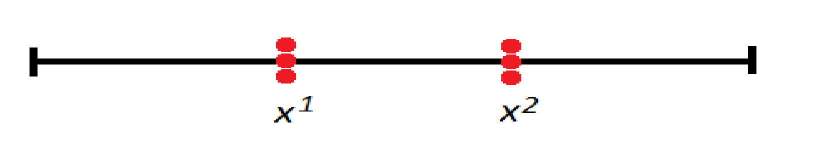
- E.g., Borda, again: $s = (m-1, m-2, \ldots, 2, 1, 0)$.
- Rules such as $s = (2, 1, \ldots, 1, 0)$ or $s = (6, 4, 4, 2, 2, 0)$.

Bipositional symmetric NCNE

What about the following kind of NCNE when m is even?



Theorem (C.-S. 2011) Suppose m is even. Then the profile with half the candidates at $0 < x^1 < 1/2$ and half at $x^2 = 1 - x^1$, is in NCNE if and only if both

$$\frac{s_{m/2} + s_{m/2+1}}{2} < \bar{s} \tag{2}$$

and

$$\frac{s_1 + s_{m/2} - 2\bar{s}}{2(s_1 - s_{m/2+1})} \leq x^1 \leq \frac{2\bar{s} - s_m - s_{m/2}}{2(s_1 - s_{m/2})}. \tag{3}$$

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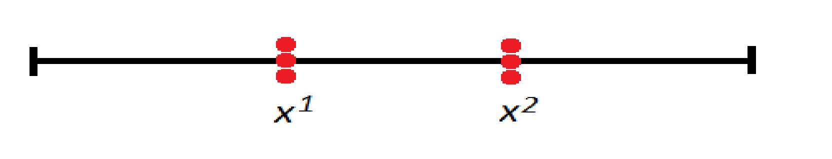

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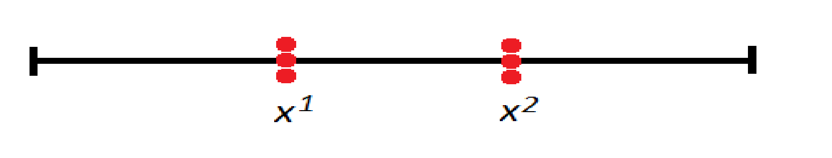

- When $m \geq 6$ there exist best-rewarding, intermediate and worst-punishing rules that allow NCNE by the theorem.
- E.g., the rules $s = (4, 3, 1, 1, 0, 0)$, $s = (10, 10, 4, 3, 3, 0)$ and $s = (2, 2, 1, 1, 1, 0)$ are best-rewarding, intermediate and worst-punishing respectively.

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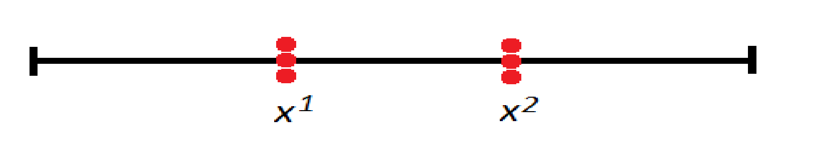

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- Hence, the dichotomy between NCNE and CNE when $m = 4, 5$ does not extend.

Multipositional NCNE

Theorem (C.-S. 2011). Let there be $m = qr$ candidates, $q \geq 2$. Let

$$s = (s_1, \dots, s_{r-1}, 0, \mid \underbrace{0, \dots, 0}_{r} \mid \cdots \mid \underbrace{0, \dots, 0}_{r})$$

be a scoring rule. Divide the interval into q equally sized subintervals. Then the profile in which r candidates locate at the half-way point of each subinterval is a NCNE if and only if $c(s', r) \leq 1/2$, where $s' = (s_1, \dots, s_{r-1}, 0)$.

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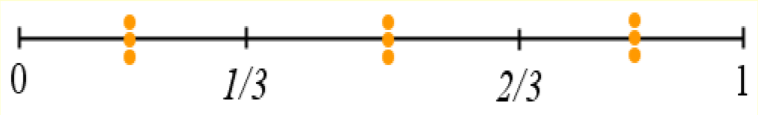

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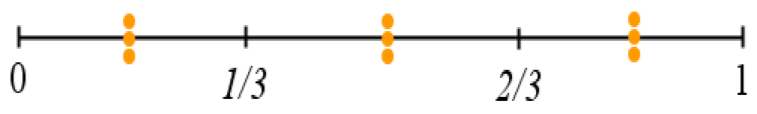

Example: $m = 9$ candidates and $q = r = 3$. NCNE for rules: 2-approval, $s = (1, 1, 0, 0, 0, 0, 0, 0, 0)$ or $(2, 1, 0, 0, 0, 0, 0, 0, 0)$.

Conclusions

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- For worst-punishing rules (when $c(s, m) \leq 1/2$), convergent equilibria are always possible. When $m = 4, 5$, NCNE cannot exist, though this dichotomy does not extend to $m \geq 6$.

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But $c(s, m)$ is not the only important factor: convexity and the way the scores are decreasing is also critical.

- Most concave rules, symmetric rules, and rules with sufficiently repeated highest scores all do not allow NCNE.

Future research

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- Develop an algorithm of deciding if a scoring rules has NCNE. Investigate complexity.
- What would happen in a dynamic situation? Would candidates converge to the equilibria? Will any clustering be observed if no equilibria exist?