Truthful Cake Cutting

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Mechanism 1 (Cut and Choose)

Agent one divides the cakes into two pieces they consider equal. Agent two is given a piece of their choice. Agent one is given the remaining piece.

The Cake Cutting Situation

Three components:

- The cake: the unit interval, $[0, 1]$.
- A set of n agents with utility functions over the cake.
- A mechanism that effects an allocation of the cake among the agents.

Modelling Taste

Agent i is associated with a utility function u_i satisfying:

- Normalisation: $u_i([0,1]) = 1$.
- Additivity: $u_i(X \cup Y) = u_i(X) + u_i(Y)$ for disjoint X, Y.
- Non-atomicity: $u_i([a,a]) = 0$.
- Non-negativity: $u_i(X) \geq 0$.

This is usually achieved with the aid of a density function, t_i:

$$u_i([a,b]) = \int_a^b t_i(x)\ dx$$

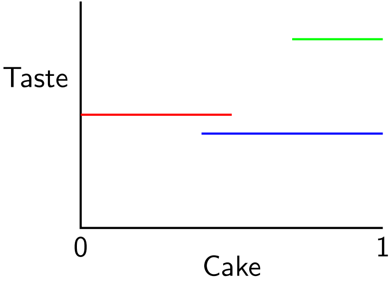
Restricted Preferences

We consider the case of piecewise uniform preferences.

- P_i : the intervals of the cake which agent i values.
- For computational purposes, we require that the endpoints be rational.
- $u_i(X) = \frac{length(X \cap P_i)}{length(P_i)}$.

Running Example

$\{P_1, P_2, P_3\} = \{[0, 0.5], [0.4, 1], [0.7, 1]\}$



The Mechanism

A mechanism is a function mapping an n-tuple of strategies, $(S_1, \ldots, S_n)$, to an allocation, $A = (A_1, \ldots, A_n)$ where A_i, A_j are portions: disjoint subsets of the cake.

In general there is no requirement for this function to be computable.

Non-Effective Cake Cutting

Theorem 1 ([a])

In any cake cutting situation, there exists an allocation $A = (A_1, \ldots, A_n)$ *such that* $u_i(A_j) = 1/n$ *for all* i, j.

[a] N. Alon. Splitting Necklaces. Advances in Mathematics, 63(3):247-253, 1987

Mechanism 2 ([b])

Find such an allocation. Randomly assign the portions to the agents.

[b] E. Mossel and O. Tamuz. Truthful fair division. Proceedings of SAGT'10

Properties of Allocations

Criteria of equity:

- Proportionality: $u_i(A_i) \geq 1/n$ for all i.
- Envy freeness: $u_i(A_i) \geq u_i(A_j)$ for all i, j.
- Equitability: $u_i(A_i) = u_j(A_j)$ for all i, j.

Criteria of efficiency:

- Don't throw the cake away.
- Non-wastefulness: if $u_i(X) = 0$ then $X \subseteq A_i$ only if $u_j(X) = 0$ for all j.
- Pareto efficiency: There is no allocation A' such that $u_i(A'_i) \geq u_i(A_i)$ for all i and $u_j(A'_j) > u_j(A_j)$ for some j. In the case of piecewise uniform preferences this is equivalent to non-wastefulness.

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The standard approach is to consider "weak truthfulness": a mechanism is proportional or envy free if every agent's sincere strategy guarantees their portion to be proportional or envy free, regardless of the strategies played by the other agents. This approach makes little sense with efficiency criteria which are global (what does it mean for a single portion to be Pareto efficient?).

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We consider instead properly truthful mechanisms, where it is in an agent's best interest to play a sincere strategy ($S_i = P_i$).

The Importance of Being Earnest

Why truthful mechanisms?

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Why truthful mechanisms?

Theorem 2 (Revelation Principle)

For every mechanism with dominant strategy equilibria there exists a truthful mechanism with the same equilibria.

Dictatorial Cake Cutting

Mechanism 3 (Lex Order)

Form a linear order $\prec$ over the agents. Allocate agent i:

$$A_i = S_i \setminus \bigcup_{j \prec i} S_j$$

Proposition 1

Lex Order is truthful and non-wasteful.

$\{P_1, P_2, P_3\} = \{[0, 0.5], [0.4, 1], [0.7, 1]\}$

Let $1 \prec 2 \prec 3$.

$A_1 = [0, 0.5]$
$A_2 = [0.5, 1]$
$A_3 = \emptyset$

Mechanism 4 ([a])

Let $\mathfrak{X}$ be a subset of the cake and $\mathfrak{A}$ a subset of the agents. Let $D(\mathfrak{A}, \mathfrak{X})$ be the length of all the intervals of $\mathfrak{X}$ valued by at least one agent in $\mathfrak{A}$. Define:

$$avg(\mathfrak{A}, \mathfrak{X}) = \frac{D(\mathfrak{A}, \mathfrak{X})}{\#\mathfrak{A}}$$

Find a subset of the agents, $\mathfrak{A}_1$, such that $avg(\mathfrak{A}_1, [0, 1])$ is minimised. Allocate every agent in $\mathfrak{A}_1$ a slice of length $avg(\mathfrak{A}_1, [0, 1])$ consisting only of intervals the agent values. Recurse on the remaining agents and the remaining cake.

[a] Y. Chen, J. K. Lai, D. C. Parkes, and A. D. Procaccia. Truth, justice and cake cutting. Proceedings of COMSOC 2010

$$\{P_1, P_2, P_3\} = \{[0, 0.5], [0.4, 1], [0.7, 1]\}$$

$$\begin{aligned}
avg(\{1\}, [0,1]) &= 0.5 \\
avg(\{2\}, [0,1]) &= 0.6 \\
avg(\{3\}, [0,1]) &= \color{red}{0.3} \\
avg(\{1,2\}, [0,1]) &= 0.5 \\
avg(\{1,3\}, [0,1]) &= 0.4 \\
avg(\{2,3\}, [0,1]) &= \color{red}{0.3} \\
avg(\{1,2,3\}, [0,1]) &= 0.\dot{3}
\end{aligned}$$

Divide $[0.4, 1]$ between 2 and 3.

$A_2 = [0.4, 0.7]$
$A_3 = [0.7, 1]$

$$avg(\{1\}, [0, 0.4]) \quad = \quad 0.4$$

Divide $[0, 0.4]$ between 1.

$A_1 = [0, 0.4]$

Strategic Cake Cutting

Mechanism 5 (Length Game)

Form a linear order $\prec$ over the agents such that $i \prec j$ if $length(S_i) < length(S_j)$. If $length(S_i) = length(S_j)$ set $i \prec j$ or $j \prec i$ arbitrarily. Allocate agent i:

$$A_i = S_i \setminus \bigcup_{j \prec i} S_j$$

Proposition 2

The equilibria of Length Game are payoff-equivalent to the allocations of Mechanism 4.

An allocation is a Length Game equilibrium if and only if:

- $S_i \subseteq P_i$.
- $\bigcup P_i \subseteq \bigcup S_i$.
- If $S_j \cap P_i \neq \emptyset$ then $length(S_j) \leq length(S_i)$.

We can infer that an equilibrium is non-wasteful and envy-free.

With $\{P_1, P_2, P_3\} = \{[0, 0.5], [0.4, 1], [0.7, 1]\}$,
$(S_1, S_2, S_3) = ([0, 0.4], [0.4, 0.7], [0.7, 1])$ is an equilibrium profile.

$[0, 0.4] \subseteq [0, 0.5]$
$[4, 0.7] \subseteq [0.4, 1]$
$[0.7, 1] \subseteq [0.7, 1]$

$\bigcup P_i = 1 = \bigcup S_i$

$S_2 \cap P_1 \neq \emptyset$, $0.3 \leq 0.4$
$S_3 \cap P_2 \neq \emptyset$, $0.3 \leq 0.3$

Hierarchical Cake Cutting

Mechanism 6

Let $\preceq$ be a partial order over the agents. Define a *tier* to be a maximal subset of agents such that $a \preceq b$ and $b \preceq a$ for all a, b in the tier. The agents in the top tier divide all cake valued by at least one agent in the tier amongst themselves using Length Game. Recurse on the remaining agents and the remaining cake.

Proposition 3

Mechanism 6 has equilibria and is non-wasteful. Lex Order is Mechanism 6 where $\preceq$ is total. Length Game is Mechanism 6 where $\preceq$ is an equivalence relation.

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Conjecture 1

All truthful, non-wasteful cake cutting mechanisms are payoff-equivalent to instances of Mechanism 6.

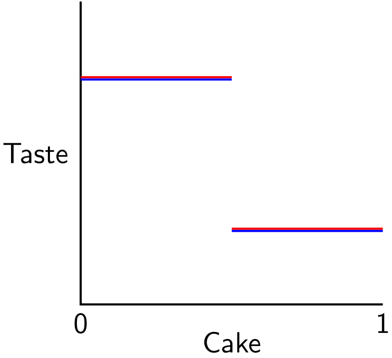
Piecewise Constant Preferences

We have hereto dealt with piecewise uniform preferences. Consider the more general piecewise constant preferences, where an agent's density function is some normalised step function.

$P_i =$ all the intervals agent i has non-zero density over, $P_i^q =$ the intervals agent i has density q over.

Equilibrium must satisfy:

- $S_i \subseteq P_i$.
- $\bigcup P_i \subseteq \bigcup S_i$.
- If $S_j \cap P_i \neq \emptyset$ then $length(S_j) \leq length(S_i)$.
- If $P_i^q \cap P_j \neq \emptyset$ and $P_i^r \cap P_j \neq \emptyset$ and $q > r$ then $S_i \cap (P_i^r \cap P_j) \neq \emptyset$ only if $S_j \cap (P_i^q \cap P_j) \neq \emptyset$.


Taste
0
1
Cake

The Cake is a Lie?

If the conjecture holds, it therefore follows that Lex Order is the only truthful, non-wasteful mechanism.

Cake cutting is either inherently wasteful or deceitful.