Power measures derived from the sequential query process

Mark C. Wilson (with R. Reyhani and G. Pritchard)
www.cs.auckland.ac.nz/~mcw/blog/

Department of Computer Science
University of Auckland

3rd CMSS Summer Workshop, 2012-02-21

The University of Auckland
NEW ZEALAND

Key references

LV2008 A. Laruelle, F. Valenciano. Voting and Collective Decision-Making. Cambridge University Press, 2008.

Key references

LV2008 A. Laruelle, F. Valenciano. Voting and Collective Decision-Making. Cambridge University Press, 2008.

DNW1981 P. Dubey, A. Neyman, R. J. Weber. Value theory without efficiency. Mathematics of Operations Research 6 (1981), 122–128.

Key references

LV2008 A. Laruelle, F. Valenciano. Voting and Collective Decision-Making. Cambridge University Press, 2008.

DNW1981 P. Dubey, A. Neyman, R. J. Weber. Value theory without efficiency. Mathematics of Operations Research 6 (1981), 122–128.

BEF2011 Y. Bachrach, E. Elkind, P. Faliszewski. Coalitional Voting Manipulation: A Game-Theoretic Perspective. IJCAI 2011: 49-54

TU games

- A transferable utility (TU) game $G = (X, v)$ is specified by a finite set X (the players) and a characteristic function $v : 2^X \to \mathbb{R}$.

TU games

- A transferable utility (TU) game $G = (X, v)$ is specified by a finite set X (the players) and a characteristic function $v : 2^X \to \mathbb{R}$.
- Usually, $v(\emptyset) = 0$ is required - we do here.

TU games

- A transferable utility (TU) game $G = (X, v)$ is specified by a finite set X (the players) and a characteristic function $v : 2^X \to \mathbb{R}$.
- Usually, $v(\emptyset) = 0$ is required - we do here.
- For each $S \subseteq X$, $v(S)$ is supposed to represent the total payoff to the coalition S.

TU games

- A transferable utility (TU) game $G = (X, v)$ is specified by a finite set X (the players) and a characteristic function $v : 2^X \to \mathbb{R}$.
- Usually, $v(\emptyset) = 0$ is required - we do here.
- For each $S \subseteq X$, $v(S)$ is supposed to represent the total payoff to the coalition S.
- Notation:

TU games

- A transferable utility (TU) game $G = (X, v)$ is specified by a finite set X (the players) and a characteristic function $v : 2^X \to \mathbb{R}$.
- Usually, $v(\emptyset) = 0$ is required - we do here.
- For each $S \subseteq X$, $v(S)$ is supposed to represent the total payoff to the coalition S.
- Notation:
  - $\mathcal{G}$: collection of all TU games - a real vector space

TU games

- A transferable utility (TU) game $G = (X, v)$ is specified by a finite set X (the players) and a characteristic function $v : 2^X \to \mathbb{R}$.
- Usually, $v(\emptyset) = 0$ is required - we do here.
- For each $S \subseteq X$, $v(S)$ is supposed to represent the total payoff to the coalition S.
- Notation:
  - $\mathcal{G}$: collection of all TU games - a real vector space
  - $\mathcal{G}(X)$: collection of all TU games with underlying set X - a real vector space

Simple games

- A simple game is a TU game with $v(S) \in \{0, 1\}$ for all $S \subseteq X$.

Simple games

- A simple game is a TU game with $v(S) \in \{0, 1\}$ for all $S \subseteq X$.
- Alternatively, it can be specified by X and the set of winning coalitions, $W := \{S \subseteq X \mid v(S) = 1\}$.

Simple games

- A simple game is a TU game with $v(S) \in \{0, 1\}$ for all $S \subseteq X$.
- Alternatively, it can be specified by X and the set of winning coalitions, $W := \{S \subseteq X \mid v(S) = 1\}$.
- Often, $\emptyset \notin W$ is required - we do here. Often $W \neq \emptyset$ is required - we don't.

Simple games

- A simple game is a TU game with $v(S) \in \{0, 1\}$ for all $S \subseteq X$.
- Alternatively, it can be specified by X and the set of winning coalitions, $W := \{S \subseteq X \mid v(S) = 1\}$.
- Often, $\emptyset \notin W$ is required - we do here. Often $W \neq \emptyset$ is required - we don't.
- Notation:

Simple games

- A simple game is a TU game with $v(S) \in \{0,1\}$ for all $S \subseteq X$.
- Alternatively, it can be specified by X and the set of winning coalitions, $W := \{S \subseteq X \mid v(S) = 1\}$.
- Often, $\emptyset \notin W$ is required - we do here. Often $W \neq \emptyset$ is required - we don't.
- Notation:
    - $\mathcal{SG}$: collection of all simple games - a lattice

Simple games

- A simple game is a TU game with $v(S) \in \{0, 1\}$ for all $S \subseteq X$.
- Alternatively, it can be specified by X and the set of winning coalitions, $W := \{S \subseteq X \mid v(S) = 1\}$.
- Often, $\emptyset \notin W$ is required - we do here. Often $W \neq \emptyset$ is required - we don't.
- Notation:
  - $\mathcal{SG}$: collection of all simple games - a lattice
  - $\mathcal{SG}(X)$: collection of all simple games with underlying set X - a lattice

Key motivating examples of simple games

- Unanimity games: choose $S \subseteq X$ and let $v(T) = 1$ if and only if $S \subseteq T$. Every member of S has a veto.

Key motivating examples of simple games

- Unanimity games: choose $S \subseteq X$ and let $v(T) = 1$ if and only if $S \subseteq T$. Every member of S has a veto.
- Weighted majority games $[q; w_1, w_2, \ldots w_n]$. Player i has weight w_i; choose a quota q and let $v(S) = 1$ iff $\sum_{i \in S} w_i \geq q$. Used to model yes-no voting in committees. Examples: stockholder elections, EU Council of Ministers, ordinary majority voting in Parliament.

Key motivating examples of simple games

- Unanimity games: choose $S \subseteq X$ and let $v(T) = 1$ if and only if $S \subseteq T$. Every member of S has a veto.
- Weighted majority games $[q; w_1, w_2, \ldots w_n]$. Player i has weight w_i; choose a quota q and let $v(S) = 1$ iff $\sum_{i \in S} w_i \geq q$. Used to model yes-no voting in committees. Examples: stockholder elections, EU Council of Ministers, ordinary majority voting in Parliament.
- Disequilibrium games: for a given noncooperative game and fixed profile of actions, declare a subset to be winning if is a witness to the profile not being a strong Nash equilibrium. Examples: voting rules with the sincere profile.

Basic concepts of TU games and simple games

monotonicity $S \subseteq T \implies v(S) \leq v(T)$.

We usually assume monotonicity for simple games, in which case we need only specify the minimal winning coalitions in order to specify the game. A dummy is not an element of any minimal winning coalition.

Basic concepts of TU games and simple games

monotonicity $S \subseteq T \implies v(S) \leq v(T)$.

dummy $i \in X$ is a dummy if for all $S \subseteq X$, $v(S) = v(S \setminus \{i\})$.

We usually assume monotonicity for simple games, in which case we need only specify the minimal winning coalitions in order to specify the game. A dummy is not an element of any minimal winning coalition.

Basic concepts of TU games and simple games

monotonicity $S \subseteq T \implies v(S) \leq v(T)$.

dummy $i \in X$ is a dummy if for all $S \subseteq X$,
$v(S) = v(S \setminus \{i\})$.

proper simple game $S \in W, T \in W \implies S \cap T \neq \emptyset$.

We usually assume monotonicity for simple games, in which case we need only specify the minimal winning coalitions in order to specify the game. A dummy is not an element of any minimal winning coalition.

Example: voting in the EU Council of Ministers

- From 1958 to 2003 various weighted majority games were used. Such games are proper if the quota is large enough (more than half the sum of all weights).

Example: voting in the EU Council of Ministers

- From 1958 to 2003 various weighted majority games were used. Such games are proper if the quota is large enough (more than half the sum of all weights).
- In first version (Treaty of Rome), game was $[12; 4, 4, 4, 2, 2, 1]$. Luxembourg was a dummy!

Example: voting in the EU Council of Ministers

- From 1958 to 2003 various weighted majority games were used. Such games are proper if the quota is large enough (more than half the sum of all weights).
- In first version (Treaty of Rome), game was $[12; 4, 4, 4, 2, 2, 1]$. Luxembourg was a dummy!
- Last version (1995–2003) had $[62; 10, 10, 10, 10, 8, 5, 5, 5, 5, 4, 4, 3, 3, 3, 2]$.

Example: voting in the EU Council of Ministers

- From 1958 to 2003 various weighted majority games were used. Such games are proper if the quota is large enough (more than half the sum of all weights).
- In first version (Treaty of Rome), game was $[12; 4, 4, 4, 2, 2, 1]$. Luxembourg was a dummy!
- Last version (1995–2003) had $[62; 10, 10, 10, 10, 8, 5, 5, 5, 5, 4, 4, 3, 3, 3, 2]$.
- Treaty of Nice (currently in force) uses weights (totalling 345) but has more conditions. A coalition is winning iff it has at least 50% of the countries, 74% of the weights, 62% of the population.

Example: voting in the EU Council of Ministers

- From 1958 to 2003 various weighted majority games were used. Such games are proper if the quota is large enough (more than half the sum of all weights).
- In first version (Treaty of Rome), game was $[12; 4, 4, 4, 2, 2, 1]$. Luxembourg was a dummy!
- Last version (1995–2003) had $[62; 10, 10, 10, 10, 8, 5, 5, 5, 5, 4, 4, 3, 3, 3, 2]$.
- Treaty of Nice (currently in force) uses weights (totalling 345) but has more conditions. A coalition is winning iff it has at least 50% of the countries, 74% of the weights, 62% of the population.
- Treaty of Lisbon (from 2014): coalition wins iff it has at least 55% of countries and 65% of population. This method is easily implemented if new members join, and avoids complex negotiations over weights.

The basic model

- Let (X, W) be a simple game, where $|X| = n$.

The basic model

- Let (X, W) be a simple game, where $|X| = n$.
- Choose (query) elements one at a time uniformly without replacement, until a winning coalition is found.

The basic model

- Let (X, W) be a simple game, where $|X| = n$.
- Choose (query) elements one at a time uniformly without replacement, until a winning coalition is found.
- This is the same process considered by Shapley and Shubik in defining their famous power index.

The basic model

- Let (X, W) be a simple game, where $|X| = n$.
- Choose (query) elements one at a time uniformly without replacement, until a winning coalition is found.
- This is the same process considered by Shapley and Shubik in defining their famous power index.
- Let Q be the random variable equal to the number of queries in this process, and $\overline{Q}$ its expectation.

The basic model

- Let (X, W) be a simple game, where $|X| = n$.
- Choose (query) elements one at a time uniformly without replacement, until a winning coalition is found.
- This is the same process considered by Shapley and Shubik in defining their famous power index.
- Let Q be the random variable equal to the number of queries in this process, and $\overline{Q}$ its expectation.
- If no winning coalition exists, let Q take the value $n + 1$.

Another interpretation of $\overline{Q}$

- For $k \in \mathbb{N}$, define the probability measure m_k to be the uniform measure on the set of all subsets of X of size k, and let W_k be the set of winning coalitions of size k.

Another interpretation of $\overline{Q}$

- For $k \in \mathbb{N}$, define the probability measure m_k to be the uniform measure on the set of all subsets of X of size k, and let W_k be the set of winning coalitions of size k.
- For each k with $0 \leq k \leq n$,

$$\Pr(Q \leq k) = \Pr(W_k)$$

where the latter probability is with respect to m_k.

Another interpretation of $\overline{Q}$

- For $k \in \mathbb{N}$, define the probability measure m_k to be the uniform measure on the set of all subsets of X of size k, and let W_k be the set of winning coalitions of size k.
- For each k with $0 \leq k \leq n$,

$$\Pr(Q \leq k) = \Pr(W_k)$$

  where the latter probability is with respect to m_k.
- In other words, the probability that we require at most k queries to find a winning coalition equals the probability that a uniformly randomly chosen k-subset is a winning coalition.

Another interpretation of $\overline{Q}$

- For $k \in \mathbb{N}$, define the probability measure m_k to be the uniform measure on the set of all subsets of X of size k, and let W_k be the set of winning coalitions of size k.
- For each k with $0 \leq k \leq n$,

$$\Pr(Q \leq k) = \Pr(W_k)$$

where the latter probability is with respect to m_k.
- In other words, the probability that we require at most k queries to find a winning coalition equals the probability that a uniformly randomly chosen k-subset is a winning coalition.
- By a standard computation involving tail probabilities, we have

$$\overline{Q} = n + 1 - \sum_{k=0}^{n} \frac{|W_k|}{\binom{n}{k}}.$$

Changing variables

- Let $F : \mathbb{N}^2 \to \mathbb{R}$. Say F is an admissible change of variables if $F(n, \cdot)$ is decreasing, $F(n, 0) = 1$ and $F(n, k) = 0$ whenever $k > n$.

Changing variables

- Let $F : \mathbb{N}^2 \to \mathbb{R}$. Say F is an admissible change of variables if $F(n, \cdot)$ is decreasing, $F(n, 0) = 1$ and $F(n, k) = 0$ whenever $k > n$.
- There is a bijection $F \leftrightarrow f$ given by

$$f(n, k) = \frac{F(n, k) - F(n, k+1)}{\binom{n}{k}}$$

Note that F is admissible if and only if f is nonnegative and $\sum_{k=0}^{n} f(n, k)\binom{n}{k} = 1$.

The University of Auckland
NEW ZEALAND

Changing variables

- Let $F : \mathbb{N}^2 \to \mathbb{R}$. Say F is an admissible change of variables if $F(n, \cdot)$ is decreasing, $F(n, 0) = 1$ and $F(n, k) = 0$ whenever $k > n$.
- There is a bijection $F \leftrightarrow f$ given by

$$f(n, k) = \frac{F(n, k) - F(n, k+1)}{\binom{n}{k}}$$

  Note that F is admissible if and only if f is nonnegative and $\sum_{k=0}^{n} f(n, k)\binom{n}{k} = 1$.
- There is a bijection $F \leftrightarrow \mu$ given by

$$\mu(n, j) = F(n, k) - F(n, k+1)$$

  Note that F is admissible if and only if for each n, $\mu(n, \cdot)$ is a probability measure on $\{0, \ldots, n\}$.

Q_F^*

- Define $Q_F^* : \mathcal{SG} \to \mathbb{R}$ by

$$Q_F^*(G) = E[F(Q)]$$

where the expectation is taken as above.

Q_F^*

- Define $Q_F^* : \mathcal{SG} \to \mathbb{R}$ by

$$Q_F^*(G) = E[F(Q)]$$

where the expectation is taken as above.

- We have

$$Q_F^* = \sum_{k=0}^{n} f(n,k)|W_k|$$

The University of Auckland NEW ZEALAND

Q^*_F

- Define $Q^*_F : \mathcal{SG} \to \mathbb{R}$ by

$$Q^*_F(G) = E[F(Q)]$$

where the expectation is taken as above.

- We have

$$Q^*_F = \sum_{k=0}^{n} f(n,k)|W_k|$$

- There is an obvious generalization to TU-games:

$$Q^*_F(G) = \sum_{k=0}^{n} f(n,k) \sum_{|S|=k, S\subseteq X} v(S) = \sum_{S\subseteq X} f(n,|S|)v(S).$$

Properties of Q_F^*

- Q_F^* is a decisiveness index on $\mathcal{SG}(X)$. It satisfies:

Properties of Q_F^*

- Q_F^* is a decisiveness index on $\mathcal{SG}(X)$. It satisfies:
  - Anonymity: depends only on the isomorphism class of the game.

Properties of Q_F^*

- Q_F^* is a decisiveness index on $\mathcal{SG}(X)$. It satisfies:
  - Anonymity: depends only on the isomorphism class of the game.
  - Positivity: is nonnegative on monotone games.

Properties of Q_F^*

- Q_F^* is a decisiveness index on $\mathcal{SG}(X)$. It satisfies:
  - Anonymity: depends only on the isomorphism class of the game.
  - Positivity: is nonnegative on monotone games.
  - Dummy: adding a null player (not a member of any winning coalition) has no effect.

Properties of Q_F^*

- Q_F^* is a decisiveness index on $\mathcal{SG}(X)$. It satisfies:
  - Anonymity: depends only on the isomorphism class of the game.
  - Positivity: is nonnegative on monotone games.
  - Dummy: adding a null player (not a member of any winning coalition) has no effect.
  - Regularity: is strictly positive unless the game has no winning coalitions.

Properties of Q_F^*

- Q_F^* is a decisiveness index on $\mathcal{SG}(X)$. It satisfies:
  - Anonymity: depends only on the isomorphism class of the game.
  - Positivity: is nonnegative on monotone games.
  - Dummy: adding a null player (not a member of any winning coalition) has no effect.
  - Regularity: is strictly positive unless the game has no winning coalitions.
- Special cases:

Properties of Q_F^*

- Q_F^* is a decisiveness index on $\mathcal{SG}(X)$. It satisfies:
  - Anonymity: depends only on the isomorphism class of the game.
  - Positivity: is nonnegative on monotone games.
  - Dummy: adding a null player (not a member of any winning coalition) has no effect.
  - Regularity: is strictly positive unless the game has no winning coalitions.
- Special cases:
  - Choosing $f(n, k) = 2^{-n}$ yields the Coleman index.

Properties of Q_F^*

- Q_F^* is a decisiveness index on $\mathcal{SG}(X)$. It satisfies:
  - Anonymity: depends only on the isomorphism class of the game.
  - Positivity: is nonnegative on monotone games.
  - Dummy: adding a null player (not a member of any winning coalition) has no effect.
  - Regularity: is strictly positive unless the game has no winning coalitions.
- Special cases:
  - Choosing $f(n, k) = 2^{-n}$ yields the Coleman index.
  - For self-dual (strong and proper) games, $Q_F^* = 1/2$.

The University of Auckland
NEW ZEALAND

Properties of Q_F^*

- Q_F^* is a decisiveness index on $\mathcal{SG}(X)$. It satisfies:
  - Anonymity: depends only on the isomorphism class of the game.
  - Positivity: is nonnegative on monotone games.
  - Dummy: adding a null player (not a member of any winning coalition) has no effect.
  - Regularity: is strictly positive unless the game has no winning coalitions.
- Special cases:
  - Choosing $f(n, k) = 2^{-n}$ yields the Coleman index.
  - For self-dual (strong and proper) games, $Q_F^* = 1/2$.
  - For the weighted majority game with quota q, $Q_F^* = F(n, q)$.

Values and semivalues

- A value is a function $\mathcal{G} \to \mathcal{AG}$. Those satisfying Anonymity, Dummy, Positivity and Linearity are called semivalues.

Values and semivalues

- A value is a function $\mathcal{G} \to \mathcal{AG}$. Those satisfying Anonymity, Dummy, Positivity and Linearity are called semivalues.
- Dubey, Neyman and Weber (1981) showed that a value is a semivalue if and only if it has the form

$$\xi_i(G) = \sum_{k=0}^{n} p(n,k) \sum_{|S|=k, S \subseteq X} [v(S) - v(S \setminus \{i\})]$$

where $p(n,k) \geq 0$ and the following identities hold

$$\sum_k \binom{n-1}{k-1} p(n,k) = 1$$

$$p(n,k) = p(n-1,k-1) - p(n,k-1)$$

Values and semivalues

- A value is a function $\mathcal{G} \to \mathcal{AG}$. Those satisfying Anonymity, Dummy, Positivity and Linearity are called semivalues.
- Dubey, Neyman and Weber (1981) showed that a value is a semivalue if and only if it has the form

$$\xi_i(G) = \sum_{k=0}^{n} p(n,k) \sum_{|S|=k, S\subseteq X} [v(S) - v(S \setminus \{i\})]$$

where $p(n,k) \geq 0$ and the following identities hold

$$\sum_k \binom{n-1}{k-1} p(n,k) = 1$$

$$p(n,k) = p(n-1,k-1) - p(n,k-1)$$

- If all $p(n,k) \neq 0$, the semivalue is called regular.

Semivalues

- Famous semivalues include the Shapley and Banzhaf values, corresponding to $p(n,k) = [k\binom{n}{k}]^{-1}$ and $p(n,k) = 2^{1-n}$ respectively.

Semivalues

- Famous semivalues include the Shapley and Banzhaf values, corresponding to $p(n, k) = [k\binom{n}{k}]^{-1}$ and $p(n, k) = 2^{1-n}$ respectively.
- Every semivalue is uniquely determined by its value on unanimity games.

Semivalues

- Famous semivalues include the Shapley and Banzhaf values, corresponding to $p(n,k) = [k\binom{n}{k}]^{-1}$ and $p(n,k) = 2^{1-n}$ respectively.
- Every semivalue is uniquely determined by its value on unanimity games.
- Regular semivalues satisfy many nice properties, such as Young sensibility: if the marginal contribution to each S is strictly higher in one game than another, then the ξ_i have the same relation.

Semivalues

- Famous semivalues include the Shapley and Banzhaf values, corresponding to $p(n,k) = [k\binom{n}{k}]^{-1}$ and $p(n,k) = 2^{1-n}$ respectively.
- Every semivalue is uniquely determined by its value on unanimity games.
- Regular semivalues satisfy many nice properties, such as Young sensibility: if the marginal contribution to each S is strictly higher in one game than another, then the ξ_i have the same relation.
- Almost all "power measures" in the literature are semivalues. The class of probabilistic values is even more general - the coefficients p can depend on S and not just on $|S|$.

Semivalues and coalition formation models

- Consider the following model of coalition formation: fix a probability distribution on 2^X, assume that each possible coalition (subset S of X) forms with probability $p(S)$, and that only one coalition S will form.

Semivalues and coalition formation models

- Consider the following model of coalition formation: fix a probability distribution on 2^X, assume that each possible coalition (subset S of X) forms with probability $p(S)$, and that only one coalition S will form.
- The *ex ante* expected marginal contribution of i to S is

$$E[D_i(S)] := E[v(S) - v(S \setminus \{i\})] = \sum_{S : i \in S} p(S) \left(v(S) - v(S \setminus \{i\}) \right).$$

Semivalues and coalition formation models

- Consider the following model of coalition formation: fix a probability distribution on 2^X, assume that each possible coalition (subset S of X) forms with probability $p(S)$, and that only one coalition S will form.
- The *ex ante* expected marginal contribution of i to S is

$$E[D_i(S)] := E[v(S) - v(S \setminus \{i\})] = \sum_{S : i \in S} p(S)\,(v(S) - v(S \setminus \{i\}))\,.$$

- The *ex interim* expected marginal contribution of i to S, conditional on $i \in S$, is

$$\Phi_i(v,p) := E[D_i(S) \mid S \ni i] = \frac{E[D_i(S)]}{\Pr(S \ni i)}.$$

Semivalues and coalition formation models

- Consider the following model of coalition formation: fix a probability distribution on 2^X, assume that each possible coalition (subset S of X) forms with probability $p(S)$, and that only one coalition S will form.
- The *ex ante* expected marginal contribution of i to S is

$$E[D_i(S)] := E[v(S)-v(S\setminus\{i\})] = \sum_{S:i\in S} p(S)\left(v(S) - v(S \setminus \{i\})\right).$$

- The *ex interim* expected marginal contribution of i to S, conditional on $i \in S$, is

$$\Phi_i(v,p) := E[D_i(S) \mid S \ni i] = \frac{E[D_i(S)]}{\Pr(S \ni i)}.$$

- There is a bijection $p \leftrightarrow \Phi(\cdot, p)$.

Potential

- Mas-Colell and Hart (1988) introduced the idea of potential, borrowed from physics.

Potential

- Mas-Colell and Hart (1988) introduced the idea of potential, borrowed from physics.
- Let ξ be a value. A *potential* for ξ is a mapping $\Phi : \mathcal{G} \to \mathbb{R}$ such that

$$\Phi(G) - \Phi(G_{-\{i\}}) = \xi_i(G)$$

for all $G = (X, v) \in \mathcal{G}$ such that $X \neq \emptyset$. Here $G_{-\{i\}}$ is the game with player set $X \setminus \{i\}$ and the same v.

Potential

- Mas-Colell and Hart (1988) introduced the idea of potential, borrowed from physics.
- Let ξ be a value. A *potential* for ξ is a mapping $\Phi : \mathcal{G} \to \mathbb{R}$ such that

$$\Phi(G) - \Phi(G_{-\{i\}}) = \xi_i(G)$$

  for all $G = (X, v) \in \mathcal{G}$ such that $X \neq \emptyset$. Here $G_{-\{i\}}$ is the game with player set $X \setminus \{i\}$ and the same v.
- The initial condition $\Phi(\emptyset, v) = 0$ is usually assumed.

Potential

- Mas-Colell and Hart (1988) introduced the idea of potential, borrowed from physics.
- Let ξ be a value. A *potential* for ξ is a mapping $\Phi : \mathcal{G} \to \mathbb{R}$ such that

$$\Phi(G) - \Phi(G_{-\{i\}}) = \xi_i(G)$$

  for all $G = (X, v) \in \mathcal{G}$ such that $X \neq \emptyset$. Here $G_{-\{i\}}$ is the game with player set $X \setminus \{i\}$ and the same v.
- The initial condition $\Phi(\emptyset, v) = 0$ is usually assumed.
- There is a unique efficient value having a potential function, and it is the Shapley value. Explicitly, the potential looks like

$$\Phi(G) = \sum_{k=1}^{n} \frac{1}{k\binom{n}{k}} \sum_{|S|=k, S \subseteq X} v(S).$$

Potential without efficiency

- Calvo and Santos (2000) described exactly which values possess a potential function.

Potential without efficiency

- Calvo and Santos (2000) described exactly which values possess a potential function.
- The answer: ξ has a potential if and only if it satisfies Myerson's balanced contributions axiom:

$$\xi_i(G) - \xi_i(G \setminus \{j\}) = \xi_j(G) - \xi_j(G \setminus \{i\})$$

and if and only if it is path-independent.

Potential without efficiency

- Calvo and Santos (2000) described exactly which values possess a potential function.
- The answer: ξ has a potential if and only if it satisfies Myerson's balanced contributions axiom:

$$\xi_i(G) - \xi_i(G \setminus \{j\}) = \xi_j(G) - \xi_j(G \setminus \{i\})$$

  and if and only if it is path-independent.
- In particular, every semivalue has a potential function. Explicitly:

$$\Phi(G) = \sum_k p(n,k) \sum_{|S|=k} v(S)$$

The marginal function

- It is readily shown that Q^*_F is the potential function of a function q^*_F, given by

$$q^*_{F,i} = \sum_{S: i \in S} f(n, |S|) D_i(S)$$

The marginal function

- It is readily shown that Q_F^* is the potential function of a function q_F^*, given by

$$q_{F,i}^* = \sum_{S:i \in S} f(n, |S|) D_i(S)$$

- Such a function is a weighted semivalue (satisfies all properties except the normalization condition).

The marginal function

- It is readily shown that Q_F^* is the potential function of a function q_F^*, given by

$$q_{F,i}^* = \sum_{S:i \in S} f(n, |S|) D_i(S)$$

- Such a function is a weighted semivalue (satisfies all properties except the normalization condition).
- There is a bijection between probability measures on $\{0, 1, \ldots, n\}$ and weighted semivalues on $\mathcal{G}_n$ given by $\mu_n \leftrightarrow q_F^*$.

The marginal function

- It is readily shown that Q_F^* is the potential function of a function q_F^*, given by

$$q_{F,i}^* = \sum_{S:i\in S} f(n, |S|) D_i(S)$$

- Such a function is a weighted semivalue (satisfies all properties except the normalization condition).
- There is a bijection between probability measures on $\{0, 1, \ldots, n\}$ and weighted semivalues on $\mathcal{G}_n$ given by $\mu_n \leftrightarrow q_F^*$.
- Under the coalition formation model above, $q_{F,i}^*$ describes the ex ante expected contribution of i to S, while the semivalue obtained by normalizing gives the ex interim expected marginal contribution of i to S, conditional on $i \in S$.

The simplest functional form

- The choice $F(n,k) = 1 - \frac{k}{n+1}$ is the simplest form for F. It corresponds to $f(n,k) = \frac{1}{(n+1)\binom{n}{k}}$.

The simplest functional form

- The choice $F(n,k) = 1 - \frac{k}{n+1}$ is the simplest form for F. It corresponds to $f(n,k) = \frac{1}{(n+1)\binom{n}{k}}$.
- This corresponds to the coalition formation model in which we choose a coalition size uniformly, and then a coalition of that size uniformly.

The simplest functional form

- The choice $F(n,k) = 1 - \frac{k}{n+1}$ is the simplest form for F. It corresponds to $f(n,k) = \frac{1}{(n+1)\binom{n}{k}}$.
- This corresponds to the coalition formation model in which we choose a coalition size uniformly, and then a coalition of that size uniformly.
- It yields a new decisiveness index, which we call Q_0^*.

The simplest functional form

- The choice $F(n,k) = 1 - \frac{k}{n+1}$ is the simplest form for F. It corresponds to $f(n,k) = \frac{1}{(n+1)\binom{n}{k}}$.
- This corresponds to the coalition formation model in which we choose a coalition size uniformly, and then a coalition of that size uniformly.
- It yields a new decisiveness index, which we call Q_0^*.
- The sequential interpretation is that we query elements one by one until we find a winning coalition, and score 1 for each query.

Manipulability measures

- Gibbard-Satterthwaite implies that (almost) every social choice function allows a nonempty simple manipulation game for some preference profile.

Manipulability measures

- Gibbard-Satterthwaite implies that (almost) every social choice function allows a nonempty simple manipulation game for some preference profile.
- The simple game describing manipulability is complicated: can fail to be weighted, strong, proper, or nonempty.

Manipulability measures

- Gibbard-Satterthwaite implies that (almost) every social choice function allows a nonempty simple manipulation game for some preference profile.
- The simple game describing manipulability is complicated: can fail to be weighted, strong, proper, or nonempty.
- Social choice theorists have tried to measure manipulability in many ways, most of them rather crude. There has been no definition of what such a measure should be, and no desirable axioms listed.

Manipulability measures

- Gibbard-Satterthwaite implies that (almost) every social choice function allows a nonempty simple manipulation game for some preference profile.
- The simple game describing manipulability is complicated: can fail to be weighted, strong, proper, or nonempty.
- Social choice theorists have tried to measure manipulability in many ways, most of them rather crude. There has been no definition of what such a measure should be, and no desirable axioms listed.
- Measures found in the literature include: indicator of winning coalition of size 1; number of winning coalitions of size 1; minimum size of a manipulating coalition.

Manipulation measures and query model

- Idea: use a collective decisiveness measure on the associated disequilibrium game to measure the ease of manipulation of a given profile. This allows a principled choice of measure for a given situation, each rooted in a model of coalition formation.

Manipulation measures and query model

- Idea: use a collective decisiveness measure on the associated disequilibrium game to measure the ease of manipulation of a given profile. This allows a principled choice of measure for a given situation, each rooted in a model of coalition formation.
- Using the query model as above, by choosing a suitable F we can have any decisiveness index we like.

Manipulation measures and query model

- Idea: use a collective decisiveness measure on the associated disequilibrium game to measure the ease of manipulation of a given profile. This allows a principled choice of measure for a given situation, each rooted in a model of coalition formation.
- Using the query model as above, by choosing a suitable F we can have any decisiveness index we like.
- We think that Q_0 is a good candidate, because of its simplicity in terms of the sequential query model.

Manipulation measures and query model

- Idea: use a collective decisiveness measure on the associated disequilibrium game to measure the ease of manipulation of a given profile. This allows a principled choice of measure for a given situation, each rooted in a model of coalition formation.
- Using the query model as above, by choosing a suitable F we can have any decisiveness index we like.
- We think that Q_0 is a good candidate, because of its simplicity in terms of the sequential query model.
- If each voter can have a different cost to recruit (as in bribery), a TU (cost) game is more appropriate than a simple game, but similar ideas should work.

Manipulation measures and query model

- Idea: use a collective decisiveness measure on the associated disequilibrium game to measure the ease of manipulation of a given profile. This allows a principled choice of measure for a given situation, each rooted in a model of coalition formation.
- Using the query model as above, by choosing a suitable F we can have any decisiveness index we like.
- We think that Q_0 is a good candidate, because of its simplicity in terms of the sequential query model.
- If each voter can have a different cost to recruit (as in bribery), a TU (cost) game is more appropriate than a simple game, but similar ideas should work.
- Bachrach, Elkind and Faliszewski have used a closely related TU framework to study manipulation of voting rules.

Open problems

- Unify recent results on complexity and power indices (e.g. Faliszewski and coauthors) and generalize them to the case of (regular) semivalues.

Open problems

- Unify recent results on complexity and power indices (e.g. Faliszewski and coauthors) and generalize them to the case of (regular) semivalues.