On the Computation of Fully Proportional Representation

(joint work with Nadja Betzler and Johannes Uhlmann)

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We will concentrate on the multi-winner rules that solve to some extent the problem of proportional representation (PR).

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There is however a third way forward.

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The Idea of Proportional Representation

A scheme of proportional representation attempts to secure an assembly whose membership will, so far as possible, be proportionate to the volume of the different shades of political opinion held throughout the country;

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Decisions of the elected assembly will be made on the basis of their independent judgements but will be as if they reflected the will of people.

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What is the best way forward?

Dodgson's idea

Charles Dodgson (Lewis Carrol) asserted that

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The idea was further advanced by Black (1986), Chamberlin & Courant (1983) and later by Monroe (1995).

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Definition

r is a misrepresentation function if

$$\mathrm{pos}_v(c) = 1 \Longrightarrow r(v, c) = 0;$$
$$\mathrm{pos}_v(c) < \mathrm{pos}_v(c') \Longrightarrow r(v, c) \le r(v, c').$$

Positional misrepresentation function

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If $\mathbf{s} = (0, 1, 2, \ldots, m-1)$ we call it the Borda misrepresentation function and $\mathbf{s} = (0, 1, 1, \ldots, 1)$ is the approval one.

Total (Societal) Misrepresentation

By $w\colon V \to C$ we denote the function that assigns voters to representatives (or the other way around), i.e., under this assignment voter v is represented by candidate $w(v)$. The total misrepresentation of the election under w is then given by

$$\sum_{v \in V} r(v, w(v)) \quad \text{or} \quad \max_{v \in V} r(v, w(v))$$

in the classical Harsanyi's and Rawl's minimax versions.

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Mapping w respects the M-criterion if $|w(V)| = k$ and w assigns at least $\lfloor n/k \rfloor$ and at most $\lceil n/k \rceil$ voters to every candidate from $w(V)$.

Chamberlin-Courant approach

They suggested to use Borda misrepresentation function with

$$\mathbf{s} = (0, 1, 2, \ldots, m)$$

and use Harsanyi's approach to calculate the total misrepresentation. If k representatives to be elected they look for $w\colon V \to C$ such that $|w(V)| = k$ and the total misrepresentation is minimized.

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Requires weighted voting in the elected assembly.

Monroe's Fully Proportional Representation

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By using the M-criterion he avoids assigning weights to representatives in the elected assembly.

Example

Six people have to elect three representative. The profile is:

4	a	b	c	d
2	c	b	a	d

- CC-method elects $\{a^2, c\}$ with total misreprtesentation 0 (a gets weight 2, c gets weight 1);
- M-method elects $\{a, b, c\}$ with total misrepresentation 2.

Alarmingly High Complexity

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In both cases Harsanyi method of calculating the total misrepresentation was used. Can Rawlsian method help?

CC-Multiwinner Problems

CC-MULTIWINNER (CC-MW)

Given: *A set C of candidates, a multiset V of voters, a misrepresentation function r, a misrepresentation bound $R \in \mathbb{Q}_0^+$ and a positive integer k.*

Task: *Find a subset $C' \subseteq C$ of size k and an assignment of voters w such that $w(V) = C'$ and $\sum_{v \in V} r(v, w(v)) \leq R$.*

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MINIMAX CC-MULTIWINNER (MINIMAX CC-MW)

Given: *Same as in CC-Multiwinner.*

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The minimax versions of the classical Chamberlin-Courant and Monroe problems, that is Minimax CC-Multiwinner and Minimax M-Multiwinner, are also NP-complete.

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But we will see that the situation changes completely for single-peaked elections where the minimax version becomes indeed easier.

Parameterized Problems and FPT

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If $(x, k) \in \Sigma^* \times \mathbb{N}$ is an instance of a parameterized problem, we refer to x as the input and k as the parameter.

A problem P is said to be Fixed Parameter Tractable (FPT) if there is an algorithm, that given a pair $(x, k) \in \Sigma^* \times \mathbb{N}$ decides whether or not $(x, k) \in P$ in at most

$$f(k)|x|^c$$

steps, where f is an arbitrary computable function and c does not depend on k.

W-Hierarchy

There is a natural hierarchy of parameterized complexity classes

$$FPT = W[0] \subseteq W[1] \subseteq W[2] \subseteq \ldots.$$

intuitively based on the complexity of circuits required to check a solution.

Experimentally shown that W[2]-complete problems are hard even for small values of the parameter.

The Hitting Set (HS)

Several parameterized reductions in this work are from the W[2]-complete HITTING SET (HS) problem:

Given family $\mathcal{F} = \{F_1, \ldots, F_n\}$ of subsets over a universe U and an integer $k \geq 0$, decide whether there is a hitting set $U' \subseteq U$ of size at most k by which we understand a set U' such that $F_i \cap U' \neq \emptyset$ for every $1 \leq i \leq n$.

HS is NP-hard and W[2]-hard with respect to parameter k (Fellows-Downey, 1999).

The Hitting Set at work

Minimax CC-Multiwinner for $R = 0$ is exactly the HS. Let $V = V_1 \cup \ldots \cup V_m$ where V_i us the set of voters whose first preference is c_i.

> *Claim. There is a hitting set of size k for* $\mathcal{V} = \{V_1, \ldots, V_m\}$ *if and only if there is a winner set of size k for* M-MULTIWINNER *that represents all voters with total misrepresentation* $R = 0$*.*

The Table of Parameterized Complexity Results

The misrepresentation function r is either approval (A), Borda (B) or unrestricted (U).

Parameter	r	CC-MW	MINIMAX CC-MW	M-MW	MINIMAX M-MW
# win. k	A	W[2]-hard	W[2]-hard	W[2]-hard	W[2]-hard
# win. k	B	W[2]-hard	W[2]-hard	W[2]-hard	W[2]-hard
misr. R	A	NP-h for $R = 0$	NP-h for $R = 0$	NP-h for $R = 0$	NP-h for $R = 0$
misr. R	B	XP	NP-h for $R \geq 1$ P for $R = 0$	XP	NP-h for $R \geq 1$ P for $R = 0$
(R, k)	A	W[2]-hard	W[2]-hard	W[2]-hard	W[2]-hard
(R, k)	B	FPT	FPT	FPT	FPT for $R = 1$
# can.	U	FPT	FPT	FPT	FPT
# vot.	U	FPT	FPT	FPT	FPT

Results for Single-Peaked Elections

The running times depending on the number n of voters, the number m of candidates, and the number k of winners. If not stated otherwise, the result holds for an arbitrary misrepresentation function.

CC-MW	MINIMAX CC-MW	M-MW	MINIMAX M-MW
$O(nm^3)$	$O(nm)$	$O(n^5mk^3)$ for approval **?** for Borda NP-hard for integer mis. func.	$O(n^2m^2(n+m))$

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M-Multiwinner for the approval misrepresentation function for instances with a single- peaked input profile can be reduced to Max-Hard-1-RS.

MAXIMUM ONE-DIMENSIONAL RECTANGLE STABBING WITH HARD CONSTRAINTS (MAX-HARD-1-RS)

Input: *A set* $\mathcal{U} = \{u_1, \ldots, u_n\}$ *of horizontal intervals and as set* $\mathcal{S} = \{S_1, \ldots, S_m\}$ *of vertical lines with capacity* $c(S) \in \{1, \ldots, n\}$ *for every line* $S \in \mathcal{S}$*, and a positive integer* k.

Task: *Find a size-*k *set* $\mathcal{S}' \subseteq \mathcal{S}$ *and an assignment* A *with* $|A(S)| \leq c(S)$ *for each* $S \in \mathcal{S}'$ *such that* $|\bigcup_{S \in \mathcal{S}'} A(S)|$ *is maximal.*

Theorem

MAXIMUM ONE-DIMENSIONAL RECTANGLE STABBING WITH HARD CONSTRAINTS *can be solved in* $O(n^5 m k^3)$ *time.*